LECTURES 16: DIMENSIONS AND COORDINATES

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1. Dimension

There is one more term related to bases which we require, and which we have already used in the context of $\mathbb{R}^n$. This requires the following result.

Theorem. *If a vector space V has a basis with n elements, then every basis of V has exactly n elements.*

Proof. Suppose that $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are bases of V with $m > n$. Every element of the first basis can, in particular, be written as a linear combination of elements of the second basis, say

$$v_i = \sum_{j=1}^{n} c_{ij} w_j.$$

The corresponding $m \times n$ matrix C with entries $C_{ij} = c_{ij}$ cannot have linearly independent rows, since the number of pivotal columns is at most n, less than m by assumption (in particular, its transpose cannot have a pivot in each column as it has more columns than rows). Hence, writing r_i for the i-th row of C, we have a non-trivial linear combination

$$\alpha_1 r_1 + \ldots + \alpha_m r_m = 0.$$

Taking the j-th component of each side yields

$$\alpha_1 c_{1j} + \ldots + \alpha_m c_{mj} = 0.$$

Hence,

$$\alpha_1 v_1 + \ldots + \alpha_m v_m = \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} c_{ij} w_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_i c_{ij} \right) w_i = \sum_{j=1}^{n} 0 \cdot w_i = 0.$$

Thus, the v_i are linearly dependent, which is a contradiction. □

Thanks to this theorem, the following definition is sensible.

Definition. If V has a basis with $n < \infty$ elements, we say that V has **dimension** n. We also say in this case that V is **finite-dimensional**. If V has a linearly independent set with infinitely many elements, then we say V is **infinite-dimensional**.

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Example. *As expected, the dimension of* $\mathbb{R}^n$*, as well as of* F^n *for any field, is* n*. More generally, the theory of coordinates (discussed further below) implies that every finite-dimensional vector space over* F *is in some sense "the same" as* F^n*, where* n *is the dimension of the space.*

Example. *We saw that* $\emptyset$ *is a basis for* $\{0\}$*. Hence* $\{0\}$ *is a zero-dimensional vector space (which fits with our intuition that the dimension of a point should be zero).*

Example. *The typical example of infinite dimensional spaces are spaces consisting of real functions. For example, the spaces of polynomials, continuous functions, and smooth functions over* $\mathbb{R}$ *are all infinite-dimensional spaces.*

Example. *We define the* **nullity** *of a matrix,* $\mathrm{null}(A)$*, to be the dimension of its kernel (a.k.a. null space). We define its* **rank** *to be the dimension of its column space (or row space; we showed the very nice result that the dimensions one gets in either case is the same last time). Since we saw that the* **rank is the number of pivots of the matrix***, and the kernel has a basis with one element corresponding to each free column (i.e., each one without a pivot), we have already shown the famous* **rank-nullity theorem***, which states that for any matrix* A *of size* $m \times n$*,*

$$\mathrm{rk}(A) + \mathrm{null}(A) = n.$$

We will return to this fundamental theorem later, but it is nice to remark at this point that such a nice result was proven only by "simple" row reduction.

Example. *Let's find (or rather describe/count) all the subspaces of* $\mathbb{F}_2^3$*. There is only one subspace of dimension* 3*, the whole space* $\mathbb{F}_2^3$*. There is only one subspace of dimension* 0*, namely* $\{0\}$*. Any subspace of dimension one is generated by a single non-zero element. By the property* $1 + 1 = 0$ *in* $\mathbb{F}_2$*, any vector* v *satisfies* $2v = 0$*, so* $\{(0, 0, 0), v\}$ *for any non-zero vector will give a subspace of dimension* 1*. There are* 7 *such vector spaces. Finally, the subspaces of dimension* 2 *will be those spanned by two non-zero vectors. We can generate any of these by writing down two different vectors* $v_1, v_2 \neq 0$*. The corresponding subspace will be* $\{0, v_1, v_2, v_1 + v_2\}$*. There are 21 different ways to choose two different non-zero vectors from our 7 available choices, and each vector space is represented* 3 *times in a different guise, and so 7 vector spaces are generated in this manner. Overall, there are thus 16 subspaces of* $\mathbb{F}_2^3$*. One can do a similar argument in any number of dimensions. For example, there are 67 subspaces of* $\mathbb{F}_2^4$ *and* 374 *of* $\mathbb{F}_2^5$*. For large* n*, it turns out that the total number of subspaces is very nearly approximated by about* $7.3 \cdot 2^{\frac{n^2}{4}}$*.*

Example. *Using the standard basis we wrote down in a previous lecture (consisting of all possible matrices with a* 1 *in one entry and* 0*'s everywhere else), the matrix space* $M_{m\times n}(F)$ *has dimension* mn*.*

Example. *The space of polynomials of degree* $\leq n$ *has basis* $\{1, x, \dots, x^n\}$*, and hence dimension* $n + 1$*.*

Example. *Recall the field* $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. *This is a* $\mathbb{Q}$*-vector space, and has as a basis* $\{1, \sqrt{2}\}$ *(note that* $\sqrt{2}$ *isn't rational!), so in fact this field is a two-dimensional vector space over the field of rationals* $\mathbb{Q}$.

2. Coordinates

We proved earlier that if $v_1, v_2, \ldots, v_n$ is a basis of a vector space V, then for any $v \in V$ there are **unique** scalars $c_1, \ldots, c_n$ for which

$$c_1v_1 + \ldots + c_nv_n = v.$$

We call these numbers $c_1, \ldots, c_n$ the **coordinates** of v with respect to this basis, and we may occasionally refer to the vector $(c_1, \ldots, c_n)$ as the **coordinate vector** of v. As hinted at above, the utility of this result/definition is that working with vector in general vector spaces is the same as working with coordinate vectors, which are ordinary vectors in the "easier" space F^n. This, however, requires one to make a choice of a basis, and in different applications, it may not always be obvious which basis is the best.

Example. *The coordinate vector of any vector* v *in* $\mathbb{R}^n$ *with respect to the standard basis* $e_1, \ldots, e_n$, *is, basically by definition, the vector* v *itself.*

Example. *The coordinate vector of a matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2\times 2}(\mathbb{R})$ *with respect to the standard basis of* $M_{2\times 2}(\mathbb{R})$ *given by*

$$\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$$

is (a, b, c, d). *Thus, we may "identify" the space* $M_{2\times 2}(\mathbb{R})$ *with* $\mathbb{R}^4$.

Example. *Instead of the standard unit basis* $\{e_1, e_2\}$ *of* $\mathbb{R}^2$, *we may take an alternate basis* $\{(2, 3), (1, 5)\}$. *Then suppose we want to find the coordinate vector of* $(4, 7)$ *with respect to this non-standard basis. We want to solve for* $c_1, c_2 \in \mathbb{R}$ *such that* $c_1(2, 3) + c_2(1, 5) = (4, 7)$, *or equivalently*

$$\begin{cases} 2c_1 + c_2 = 4. \\ 3c_1 + 5c_2 = 7. \end{cases}$$

Solving this system gives $c_1 = 13/7$, $c_2 = 2/7$, *which are the coordinates of* $(4, 7)$ *(as is easily checked by plugging back in).*

We now describe the general procedure for finding coordinate vectors in $\mathbb{R}^n$ with respect to arbitrary bases $v_1, \ldots, v_n$. Suppose that v is a general vector in $\mathbb{R}^n$ and we wish to find its coordinates. Then we are trying to solve

$$c_1v_1 + \ldots + c_nv_n = v,$$

which is (as we have seen several times in previous lectures) a system of linear equations corresponding to finding a vector c with

$$Ac = v,$$

where A is the matrix with columns $v_1, \dots, v_n$. Of course, this is solved by finding A^{-1} and computing $c = A^{-1}v$ (or by using any methods for solving systems of linear equations which we gave before). Note that the inverse will exist, since a set of n vectors in $\mathbb{R}^n$ is a basis if and only if the determinant of A is non-zero, or equivalently, A is invertible.

Example. *If we consider the basis* $\{(1,2),(2,3)\}$ *of* $\mathbb{R}^2$*, then the coordinates of* $(5,4)$ *can be found by first taking the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

and computing its inverse (we can use the general formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix})$$

to be

$$A^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.$$

Thus the coordinate vector of $(5,4)$ *is*

$$\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \end{pmatrix}.$$

Example. *A basis for the space of polynomials* $P(x)$ *of degree at most* 3 *with* $P(3) = 0$ *is given by* $\{x-3, x^2-9, x^3-27\}$*. This is thus a three-dimensional subspace of the four-dimensional space of polynomials of degree at most* 3*. It is easy to check that* $f(x) = -x^3 + 3x^2 + 2x - 6$ *lies in this subspace. To express it in terms of this basis, we have to solve*

$$a(x-3)+b(x^2-9)+c(x^3-27) = ax+bx^2+cx^3+(-3a-9b-27c) = -x^3+3x^2+2x-6.$$

Setting powers of x *on either side equal to one another, we directly find that* $a = 2, b = 3, c = -1$ *(as well as that the constant terms are consistent with these choices), so the coordinate vector of* f *with respect to the basis above is* $(2,3,-1)$*.*