LECTURE 10: DETERMINANTS BY LAPLACE EXPANSION AND INVERSES BY ADJOINT MATRICES

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

1. Laplace expansions

By using the cofactors from the last lecture, we can find a very convenient way to compute determinants. We first give the method, then try several examples, and then discuss its proof.

Algorithm (Laplace expansion)**.** *To compute the determinant of a square matrix, do the following.*

(1) *Choose any row or column of A.*
(2) *For each element A_{ij} of this row or column, compute the associated cofactor C^{ij}.*
(3) *Multiply each cofactor by the associated matrix entry A_{ij}.*
(4) *The sum of these products is* $\det A$.

Example. *We find the determinant of*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}.$$

We make the arbitrary choice to expand along the first row. We compute the minors as

$$M^{11} = \det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}, \qquad M^{12} = \det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}, \qquad M^{13} = \det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix}.$$

Computing these 2×2 determinants, we have

$$M^{11} = 4, \qquad M^{12} = -1, \qquad M^{13} = 2.$$

By inserting signs, we find that the cofactors are

$$C^{11} = M^{11} = 4, \qquad C^{12} = -M^{12} = 1, \qquad C^{13} = M^{13} = 2.$$

Thus,

$$\det A = A_{11}C^{11} + A_{12}C^{12} + A_{13}C^{13} = 2(4) + (1) + 3(2) = 15.$$

Date: October 19, 2016.

With this method, much larger determinants are also feasible, especially when there are lots of zeros in a row or column. Generally speaking, it is a good idea when computing an example to try to expand along a row or column with as many zeros as possible, or at least with the smallest entries possible.

Example. *We find the determinant of*

$$A = \begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}.$$

We expand along the last column to find

$$\det A = 2 \det \begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5 \det \begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix}$$

We can find these two determinants by expanding them as well. For example, expanding along the first column on the first one, we find that

$$\det \begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = -2 \det \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} -3 & 2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$$
$$= -2(24) - (-24) - 0 = -48 + 24 + 0 = -24.$$

Similarly, by expanding the second 3×3 *matrix along the first column, we find that*

$$\begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = 2 \det \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} 5 & -3 \\ 6 & 4 \end{pmatrix} - \det \begin{pmatrix} 5 & -3 \\ 3 & -2 \end{pmatrix}$$
$$= 2(24) - (38) - (-1) = 11.$$

Thus, we find that

$$\det A = 2(-24) - 5(11) = -103.$$

Discussion of the proof of the algorithm. Since we know that switching two rows negates determinants (and negates the sings in the cofactors as well), and since transposes preserve determinants (meaning that we can switch the roles of rows and columns in determinant calculations), it is enough to show this when we pick the first row in the algorithm. By our definition, it is enough to show that this satisfies the 3 properties uniquely characterizing determinants. That is, if we define $f(A) = A_{11}C^{11} + \ldots + A_{1n}C^{1n}$, then we just have to show that f is multilinear in the rows of A, that it is alternating in the rows, and that $f(I_n) = 1$. The proof of multilinearity, and of the alternating property, follow from careful writing down of the objects involved, but you can also try an example and are encouraged to think through why you should believe this! For example, you are strongly encouraged to just try out these two properties on an arbitrary, say 3×3, matrix. □

2. Adjugate matrices and inverses

In addition to finding determinants quickly, we can use cofactors to quickly compute inverses of matrices. If we stick all the cofactors into a matrix, then we obtain the cofactor matrix. That is, the cofactor matrix is the matrix C such that

$$C_{ij} = C^{ij}.$$

The **adjugate matrix** (sometimes called the adjoint matrix), denoted $\mathrm{adj}(A)$, is simply the transpose of the cofactor matrix:

$$(\mathrm{adj}A)_{ij} = C^{ji}.$$

The reason this matrix is interesting is that the following result holds.

Theorem. *For any $n \times n$ matrix A, we have*

$$A \cdot \mathrm{adj}(A) = \det(A) I_n.$$

In particular, if A is invertible, then $A^{-1} = (\det A)^{-1}\mathrm{adj}(A)$.

Proof. This is essentially a restatement of the Laplace expansion algorithm above. To check it, compute the i,j-th entry of the left hand side:

$$(A \cdot \mathrm{adj}A)_{ij} = \sum_{k=1}^{n} A_{ik}(\mathrm{adj}A)_{kj} = \sum_{k=1}^{n} a_{ik}C^{jk}.$$

If $i = j$, then by Laplace expansion, we get $\det A$. If $i \neq j$, then by Laplace expansion again, we are really computing the determinant of the matrix where we replace the j-th row of A by its i-th row. But such a matrix has two rows which are the same, and hence has determinant zero. □

Example. *We continue working with the matrix*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$$

from above. We already found the first few cofactors to be

$$C^{11} = M^{11} = 4, \qquad C^{12} = -M^{12} = 1, \qquad C^{13} = M^{13} = 2.$$

Continuing in the same manner as above, we find that the matrix of cofactors is

$$\begin{pmatrix} 4 & 1 & 2 \\ 3 & 12 & -6 \\ -5 & -5 & 5 \end{pmatrix}.$$

Taking the transpose of this matrix yields that

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & 3 & -5 \\ 1 & 12 & -5 \\ 2 & -6 & 5 \end{pmatrix}.$$

Since we saw above that $\det A = 15$*, we find that*

$$A^{-1} = \begin{pmatrix} \frac{4}{15} & \frac{1}{5} & -\frac{1}{3} \\ \frac{1}{15} & \frac{4}{5} & -\frac{1}{3} \\ \frac{2}{15} & -\frac{2}{5} & \frac{1}{3} \end{pmatrix}.$$

3. Cramer's Rule

Suppose that A is invertible. Then we already know that $Ax = b$ has only one solution for any b. Of course, this solution is the vector $x = A^{-1}b$. Plugging in the adjugate yields that

$$x_j = (\det A)^{-1}(\operatorname{adj} A)b = (\det A)^{-1} \sum_{k=1}^{n} C^{kj} b_k.$$

But the sum is just a j-th column expansion of the matrix A_j obtained by replacing the j-th column with b. This gives Cramer's fomula.

Theorem (Cramer's rule)**.** *Assume the notation above. If A is invertible, then the solution to $Ax = b$ is given by*

$$x_j = \frac{\det(A_j)}{\det A}.$$

Example. *We saw above that*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$$

is invertible, with $\det A = 15$*. Thus, by Cramer's rule, the solution to* $Ax = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ *is given by*

$$x_1 = \frac{1}{15} \det \begin{pmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = -\frac{2}{15},$$

$$x_2 = \frac{1}{15} \det \begin{pmatrix} 2 & 1 & 3 \\ -1 & -2 & 1 \\ -2 & 0 & 3 \end{pmatrix} = -\frac{23}{15},$$

$$x_3 = \frac{1}{15} \det \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -2 \\ -2 & 2 & 0 \end{pmatrix} = \frac{14}{15}.$$