LECTURE 1: INTRODUCTION, VECTORS, AND DOT PRODUCTS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

1. Introduction

Firstly, I want to point out that I am here to help, so if you have a question about a lecture or the homework, don't be afraid to ask. Most likely, you aren't the only one! I am of course available by email, and you can schedule an appointment with me to meet. Furthermore, there are resources, both at TCD such as the Maths Help Center, as well as on the web (there are many great linear algebra resources online in addition to many great textbooks on the subject). In general, all information pertaining to the course, homeworks, tutorials, and notes, can be found at the course website `www.maths.tcd.ie/~lrolen/LinearAlgebra.html`.

2. What is a proof?

One of the main goals of this class is to learn how to prove things in a mathematically rigorous way. To learn what is a proof, we first have to see what's not a proof. Before stating our first fallacy, we note that $\in$ means "is an element of", and can be read as "is in", and $\mathbb{R}$ denotes the set of real numbers.

"Theorem". Every bounded, differentiable, real function of one real variable is constant.

Proof. By definition, there exist constants $M, N \in \mathbb{R}$ such that

$$M \leq f(x) \leq N$$

for all $x \in \mathbb{R}$. Differentiating both sides of both inequalities, we find that

$$0 \leq f'(x) \leq 0$$

for all x, and so f' is identically zero. This implies f is constant, as claimed. □

(The cute little box signifies the end of a proof) Obviously, this is nonsense! For instance, $\sin(x)$ is a counterexample. So the lesson is that sometimes things that formally look similar to proofs may not be proofs at all.

Watch out for fallacies.

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The last symbol on the last page denotes a curvy road warning symbol, and we will use it throughout to point out things to watch out for. We will see many examples of proof strategies throughout the course.

3. What is linear algebra?

Tl;dr: Lines, planes, and flat things.

Here is a flippant result/joke which encapsulates some common wisdom about why linear algebra is so important in mathematics.

"Theorem". All of mathematics is calculus and linear algebra.

Proof. Real life is curvy (and for example, in physics, not everything moves in a straight line at a constant speed!). However, like the Earth, everything curved locally looks flat. Using calculus, we can approximate curvy things by flat things, which are then in the domain of linear algebra. □

Sample application: You may not all remember it, but a long time ago there were several dozen competing search engines, all of which were terrible. One day, linear algebra was used to invent the page-rank algorithm, which made Google possible. The idea only uses basic linear algebra facts we will learn this semester.

4. Vectors

The heroes of this course are important objects called **vectors.** These have several incarnations, but one way to think about them is that they are "arrows" . They have two important geometric quantities associated to them: direction and magnitude. Direction is just what it sounds like; the way the arrow points, and magnitude is just the length of the arrow. Importantly, we will often think of vectors as tuples of real numbers, rather than as arrows. This is important as it gives a powerful tool to describe things algebraically (cf. the name of the course!), which is critical to work efficiently, as well as in higher-dimensional situation when we can no longer rely on drawing pictures. Comparing with the notion of arrows, we are essentially "abusing notation" to think of a tuple of numbers as either a point in Cartesian space **or** as a vector from the origin to that point; which meaning is used must be determined by context.

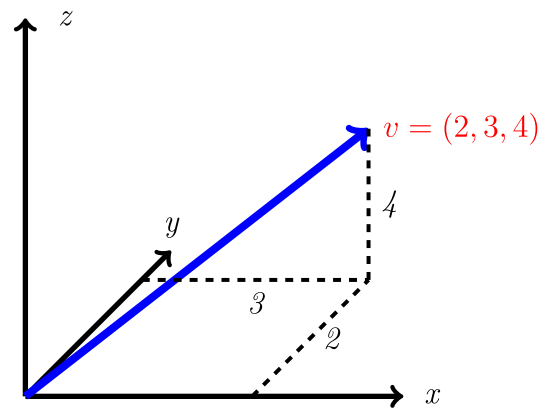
Definition. A vector is a tuple of n real numbers, such as

$$v = (v_1, v_2 \dots, v_n).$$

It is often important to work in more than 3-dimensions. This is where the power of the tuple definition comes into play; it is impossible to picture a 100-dimensional arrow but not such a stretch to imagine listing 100 numbers!

Example. *Given two points A,B, the vector* $\overrightarrow{AB}$ *is the vector pointing from A to B. One can find it algebraically by subtracting coordinates. For example, if* $A = (1, 2)$, $B = (3, 6)$, *then* $\overrightarrow{AB} = (2, 4)$.

Example. *The vector* $(2, 3, 4)$ *is the vector from the origin to* $(2, 3, 4)$, *illustrated as follows.*



4.1. **New vectors from old.** In mathematics, one is often interested in how to obtain new objects from old ones. After all, it is always a good idea to make our lives easier and not repeat mundane things over and over again. The first main operation is that of vector addition.

Definition. We define the sum of two vectors $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ to be the vector

$$v + w = (v_1 + w_1, \ldots, v_n + w_n).$$

N.B.: We will frequently use v, w for vectors and freely refer to their components using subscripts as in this definition. This is why notation is so important in mathematics: instead of restating conditions all the time, the context is already understood by our letter choices.

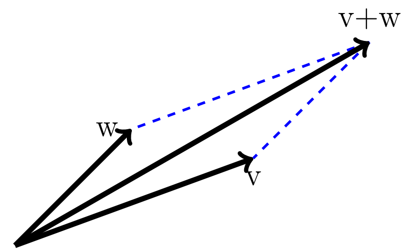
Example. *In three dimensions, we have*

$$(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

Example. *We have*

$$(1, 2) + (3, -5) = (4, -3).$$

This algebraic definition is so simple as to almost seem trivial. The real test of any definition, however, is its utility. For example, we do often think of vectors geometrically, so it is natural to ask: what is the geometric interpretation? This has a simple answer, encapsulated in the following picture of the **parallelogram rule**:



Of course, if we think of vectors as objects describing force in physics, it is not immediately obvious that this is the correct definition to correspond with reality. However, a simple set-up of pulleys can be used to perform an experiment to confirm that vector addition corresponds to addition of forces on real material objects.

The second definition is similarly straightforward on an algebraic level. It is essentially the only thing that could come to mind if you try to imagine what I mean when I tell you that you can multiply a "scalar" (just a real number) by a vector.

Definition. For a vector v and a scalar c, the scalar multiplication of c and v is

$$cv = (cv_1, \dots, cv_n).$$

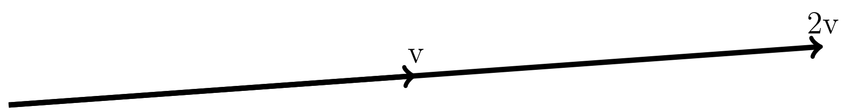
In other words, the scalar is just multiplied by each component separately.

Example. *We have*

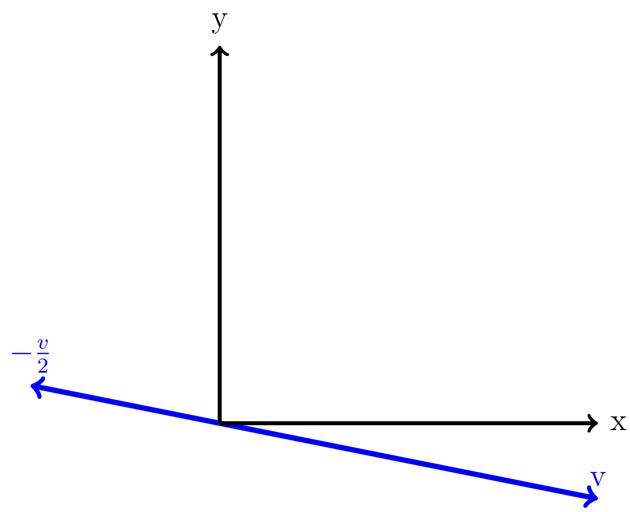
$$2(1, 2) = (2, 4).$$

As with vector addition, there is a simple geometric interpretation of this operation.

Case 1): If $c > 0$, then cv is the vector obtained by rescaling the length (magnitude) of v by a factor of c and keeping the direction the same. For example:



Case 2): If $c < 0$, then cv is the vector obtained by rescaling the length (magnitude) of v by a factor of c and reversing the direction. For example:



Case 3): If $c = 0$, then $cv = 0$.

We now come to a less obvious, but extremely important operator which takes two vectors and spits out a number.

Definition. The dot product, or scalar product, of two vectors v and w is

$$v \cdot w = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n.$$

As with vector addition, it is natural to ask what this number "means" geometrically. It turns out that it is simple to describe in terms of the lengths of v and w, as well as their (relative) directions. This fact, coupled with the nice properties below are what makes this product a staple of the industry. Moreover, if you ever see a new operator, you should always ask what properties it has, as without some of these types of properties, it will usually be unhelpful.

Theorem. *For any vectors v, w, w_1, w_2, and any $c \in \mathbb{R}$, we have the following.*

(1) $v \cdot w = w \cdot v$.
(2) $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$.
(3) $v \cdot (cw) = c(v \cdot w)$.
(4) $v \cdot v = |v|^2$, *where $|v|$ denotes the length of v (also called the magnitude or norm).*
(5) $v \cdot w = |v||w| \cos \vartheta$, *where ϑ is the angle between v and w.*

N.B. Theorem is just a fancy word for a fact (a mathematician's fact, not a politicians "fact").

We will come to the proofs of these facts next time.