L–functions for Harmonic Maass Forms

Larry Rolen

Vanderbilt University

October 22, 2023

Modular forms

- Functions on $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$.

Modular forms

- Functions on $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.
- Slash action: $f|_k\gamma(\tau) := (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d))$.

Modular forms

- Functions on $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.
- Slash action: $f|_k\gamma(\tau) := (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d))$.
- Modularity:
  1. $f|_k\gamma = f \quad \forall\gamma \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.
  2. $+$ growth conditions (classical: holomoprhic at "cusps")
  3. $+$ analytic conditions (classical: holomoprhic)

Modular forms

- Functions on $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.
- Slash action: $f|_k\gamma(\tau) := (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d))$.
- Modularity:
  1. $f|_k\gamma = f \quad \forall \gamma \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.
  2. $+$ growth conditions (classical: holomoprhic at "cusps")
  3. $+$ analytic conditions (classical: holomoprhic)
- New Quanta article: `https://tinyurl.com/vv8mcjuw`

Modular forms

- Functions on $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$.
- Slash action: $f|_k\gamma(\tau) := (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d))$.
- Modularity:
  1. $f|_k\gamma = f \quad \forall\gamma \in \Gamma \leq \text{SL}_2(\mathbb{Z})$.
  2. $+$ growth conditions (classical: holomoprhic at "cusps")
  3. $+$ analytic conditions (classical: holomoprhic)
- New Quanta article: `https://tinyurl.com/vv8mcjuw`
- "Algebra" of adjectives: weakly, quasi, meromorphic, almost...

L-functions for classical modular forms

- Cusp form $f(\tau) := \sum_{n\geq 1} a_n(f)q^n \in S_k(N) \rightsquigarrow$ *Dirichlet series*

L-functions for classical modular forms

- Cusp form $f(\tau) := \sum_{n\geq 1} a_n(f)q^n \in S_k(N) \rightsquigarrow$ *Dirichlet series*

$$L_f(s) := \sum_{n\geq 1} a_f(n)n^{-s}, \quad (\mathrm{Re}(s) \gg 0).$$

L-functions for classical modular forms

- Cusp form $f(\tau) := \sum_{n\geq 1} a_n(f)q^n \in S_k(N) \rightsquigarrow$ *Dirichlet series*

$$L_f(s) := \sum_{n\geq 1} a_f(n)n^{-s}, \quad (\mathrm{Re}(s) \gg 0).$$

- Completion: $\Lambda_f(s) := N^{s/2}(2\pi)^{-2s}\Gamma(s)L_f(s)$.

L-functions for classical modular forms

- Cusp form $f(\tau) := \sum_{n\geq 1} a_n(f)q^n \in S_k(N) \rightsquigarrow$ *Dirichlet series*

$$L_f(s) := \sum_{n\geq 1} a_f(n)n^{-s}, \quad (\mathrm{Re}(s) \gg 0).$$

- Completion: $\Lambda_f(s) := N^{s/2}(2\pi)^{-2s}\Gamma(s)L_f(s)$.
- This analytically continues to $\mathbb{C}$ with functional equation:

$$\Lambda_f(s) = \pm\Lambda_f(k-s).$$

L-functions for classical modular forms

- Cusp form $f(\tau) := \sum_{n\geq 1} a_n(f)q^n \in S_k(N) \rightsquigarrow$ *Dirichlet series*

$$L_f(s) := \sum_{n\geq 1} a_f(n)n^{-s}, \quad (\mathrm{Re}(s) \gg 0).$$

- Completion: $\Lambda_f(s) := N^{s/2}(2\pi)^{-2s}\Gamma(s)L_f(s)$.
- This analytically continues to $\mathbb{C}$ with functional equation:

$$\Lambda_f(s) = \pm\Lambda_f(k-s).$$

- If $f \in M_k(N)$ (e.g., Eisenstein series), subtract constant term.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.
- Harmonic Maass forms ($H_k(N)$): like WHMFs, but in kernel of "Laplacian" instead of holomorphic.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.
- Harmonic Maass forms ($H_k(N)$): like WHMFs, but in kernel of "Laplacian" instead of holomorphic. Splitting: $\sum_{n\gg-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n$, where $\Gamma(s,z) := \int_z^{i\infty} e^{-t}t^{s-1}dt$.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.
- Harmonic Maass forms ($H_k(N)$): like WHMFs, but in kernel of "Laplacian" instead of holomorphic. Splitting: $\sum_{n\gg-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n$, where $\Gamma(s,z) := \int_z^{i\infty} e^{-t}t^{s-1}dt$.
- HMFs have "shadows." The map $\xi_{2-k}: H_{2-k} \to S_k$ given by $2i\,\mathrm{Im}(\tau)^k\overline{\partial/\partial\overline{\tau}}$ is surjective.

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.
- Harmonic Maass forms ($H_k(N)$): like WHMFs, but in kernel of "Laplacian" instead of holomorphic. Splitting: $\sum_{n\gg-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n$, where $\Gamma(s,z) := \int_z^{i\infty} e^{-t}t^{s-1}dt$.
- HMFs have "shadows." The map $\xi_{2-k}: H_{2-k} \to S_k$ given by $2i\,\mathrm{Im}(\tau)^k\overline{\partial/\partial\overline{\tau}}$ is surjective. Finding preimages related to big problems: Lehmer's Conjecture, ranks of elliptic curves...

Forms with exponential growth

- In combinatorics, physics, enumerative geometry, often encounter functions with exp. growth at the cusps.
- Weakly holomorphic modular forms ($M_k^!(N)$): Like $M_k(N)$, but may have a "pole" at the cusps.
- Harmonic Maass forms ($H_k(N)$): like WHMFs, but in kernel of "Laplacian" instead of holomorphic. Splitting: $\sum_{n \gg -\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n$, where $\Gamma(s,z) := \int_z^{i\infty} e^{-t}t^{s-1}dt$.
- HMFs have "shadows." The map $\xi_{2-k} : H_{2-k} \to S_k$ given by $2i\,\mathrm{Im}(\tau)^k \overline{\partial/\partial\overline{\tau}}$ is surjective. Finding preimages related to big problems: Lehmer's Conjecture, ranks of elliptic curves...
- Brown's modular iterated integrals: repeated primitives of WHMFs under ∂_τ, $\partial_{\overline{\tau}}$.

L-functions for forms with exp. growth?

- For WHMFs, HMFs, etc., the poles at the cusp give (sub)-exponential growth of Fourier coefficients.

L-functions for forms with exp. growth?

- For WHMFs, HMFs, etc., the poles at the cusp give (sub)-exponential growth of Fourier coefficients.
- Bringmann-Fricke-Kent ($f \in M_k^!$): For $t_0 > 0$, define

$$L_f(s) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma\left(k-s, \frac{2\pi n}{t_0}\right)}{(2\pi n)^{k-s}}.$$

L-functions for forms with exp. growth?

- For WHMFs, HMFs, etc., the poles at the cusp give (sub)-exponential growth of Fourier coefficients.
- Bringmann-Fricke-Kent ($f \in M_k^!$): For $t_0 > 0$, define

$$L_f(s) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma\left(k-s, \frac{2\pi n}{t_0}\right)}{(2\pi n)^{k-s}}.$$

- They used this to further "explain" generalizations of Eichler-Shimura period polynomial theory to HMF theory.

L-functions for forms with exp. growth?

- For WHMFs, HMFs, etc., the poles at the cusp give (sub)-exponential growth of Fourier coefficients.
- Bringmann-Fricke-Kent ($f \in M_k^!$): For $t_0 > 0$, define

$$L_f(s) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma\left(k-s, \frac{2\pi n}{t_0}\right)}{(2\pi n)^{k-s}}.$$

- They used this to further "explain" generalizations of Eichler-Shimura period polynomial theory to HMF theory.
- No construction was known for other HMFs with exp. growth.

L-functions for forms with exp. growth?

- For WHMFs, HMFs, etc., the poles at the cusp give (sub)-exponential growth of Fourier coefficients.
- Bringmann-Fricke-Kent ($f \in M_k^!$): For $t_0 > 0$, define

$$L_f(s) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{\substack{n \gg -\infty \\ n \neq 0}} \frac{a_f(n)\Gamma\left(k - s, \frac{2\pi n}{t_0}\right)}{(2\pi n)^{k-s}}.$$

- They used this to further "explain" generalizations of Eichler-Shimura period polynomial theory to HMF theory.
- No construction was known for other HMFs with exp. growth.
- This is "symmetrized" so functional equation is "baked in."

Workaround: Use test functions

- Diamantis-Lee-Raji-R. ($f \in H_k$): Using the Laplace transform $(\mathcal{L}\varphi)(u) := \int_0^\infty e^{-ut}\varphi(t)dt$, set

$$L_f(\varphi) := \sum_{n \geq -n_0} c_f^+(n)(\mathcal{L}\varphi)(2\pi n)$$

Workaround: Use test functions

- Diamantis-Lee-Raji-R. ($f \in H_k$): Using the Laplace transform $(\mathcal{L}\varphi)(u) := \int_0^\infty e^{-ut}\varphi(t)dt$, set

$$L_f(\varphi) := \sum_{n \geq -n_0} c_f^+(n)(\mathcal{L}\varphi)(2\pi n)$$

$$+ \sum_{n<0} c_f^-(n)(-4\pi n)^{1-k} \int_0^\infty \frac{(\mathcal{L}\varphi_{2-k})(-2\pi n(2t+1))}{(1+t)^k} dt.$$

Workaround: Use test functions

- Diamantis-Lee-Raji-R. ($f \in H_k$): Using the Laplace transform $(\mathcal{L}\varphi)(u) := \int_0^\infty e^{-ut}\varphi(t)dt$, set

$$L_f(\varphi) := \sum_{n \geq -n_0} c_f^+(n)(\mathcal{L}\varphi)(2\pi n)$$

$$+\sum_{n<0} c_f^-(n)(-4\pi n)^{1-k} \int_0^\infty \frac{(\mathcal{L}\varphi_{2-k})(-2\pi n(2t+1))}{(1+t)^k} dt.$$

- Recovering previous:

$$L_f(s) = \begin{cases} L_f((2\pi)^s x^{s-1}/\Gamma(s)) & \text{if } f(\tau) \in S_k, \\ L_f(\chi_{x \geq t_0}(2\pi)^s x^{s-1}/\Gamma(s)) & \text{if } f(\tau) \in M_k^!. \end{cases}$$

Analogues of classical results

Theorem (Converse Theorem, D-L-R-R, 2023)

Suppose f has a Fourier expansion of the shape above.

Analogues of classical results

Theorem (Converse Theorem, D-L-R-R, 2023)

Suppose f has a Fourier expansion of the shape above. Set $f'(\tau) := \tau \frac{\partial f}{\partial u} + \frac{k}{2} f(\tau)$.

Analogues of classical results

Theorem (Converse Theorem, D-L-R-R, 2023)

Suppose f has a Fourier expansion of the shape above. Set $f'(\tau) := \tau\frac{\partial f}{\partial u} + \frac{k}{2}f(\tau)$. Then explicit functional equations sending $\varphi(x) \mapsto \varphi(x)|_{2-k}W_N$ under the Fricke involution $\begin{pmatrix} 0 & \sqrt{M}^{-1} \\ \sqrt{M} & 0 \end{pmatrix}$, for a finite set of Dirichlet character twisted L-functions of f and f', imply that $f \in H_k(\Gamma)$.

Analogues of classical results

Theorem (Converse Theorem, D-L-R-R, 2023)

Suppose f has a Fourier expansion of the shape above. Set $f'(\tau) := \tau\frac{\partial f}{\partial u} + \frac{k}{2}f(\tau)$. Then explicit functional equations sending $\varphi(x) \mapsto \varphi(x)|_{2-k}W_N$ under the Fricke involution $\begin{pmatrix} 0 & \sqrt{M}^{-1} \\ \sqrt{M} & 0 \end{pmatrix}$, for a finite set of Dirichlet character twisted L-functions of f and f', imply that $f \in H_k(\Gamma)$.

Theorem (Summation Formula, D-L-R-R, 2023)

If $F \in S_k$ with shadow g, then for "good" test functions φ,

$$\sum_{n \gg -\infty} c_F^+(n)\mathcal{I}(\varphi) = \sum_{n>0} \overline{a_g(n)}\mathcal{J}(\varphi).$$

Analogues of classical results

Theorem (Converse Theorem, D-L-R-R, 2023)

Suppose f has a Fourier expansion of the shape above. Set $f'(\tau) := \tau \frac{\partial f}{\partial u} + \frac{k}{2} f(\tau)$. Then explicit functional equations sending $\varphi(x) \mapsto \varphi(x)|_{2-k} W_N$ under the Fricke involution $\begin{pmatrix} 0 & \sqrt{M}^{-1} \\ \sqrt{M} & 0 \end{pmatrix}$, for a finite set of Dirichlet character twisted L-functions of f and f', imply that $f \in H_k(\Gamma)$.

Theorem (Summation Formula, D-L-R-R, 2023)

If $F \in S_k$ with shadow g, then for "good" test functions φ,

$$\sum_{n \gg -\infty} c_F^+(n) \mathcal{I}(\varphi) = \sum_{n>0} \overline{a_g(n)} \mathcal{J}(\varphi).$$

Here $\mathcal{I}$ and $\mathcal{J}$ are explicit integral transforms.

Applications

- This gives a way to sidestep “regularization” procedures.

Applications

- This gives a way to sidestep "regularization" procedures.
- Gives a method to produce operators between HMF spaces.

Applications

- This gives a way to sidestep "regularization" procedures.
- Gives a method to produce operators between HMF spaces.
- For instance, the Shimura lift was first discovered by using Converse Theroem+Rankin-Selberg.

Applications

- This gives a way to sidestep “regularization” procedures.
- Gives a method to produce operators between HMF spaces.
- For instance, the Shimura lift was first discovered by using Converse Theroem+Rankin-Selberg.
- The summation formula strongly couples “classical” MF coefficients to “mysterious” mock coefficients.

Further directions

- This gives new structure to explore. It works in other cases; October 2023: Drewitt-Pimm did for Brown's functions.

Further directions

- This gives new structure to explore. It works in other cases; October 2023: Drewitt-Pimm did for Brown's functions.

- What about eigenforms? Guerzhoy gave WHMF eigenforms.

Further directions

- This gives new structure to explore. It works in other cases; October 2023: Drewitt-Pimm did for Brown's functions.

- What about eigenforms? Guerzhoy gave WHMF eigenforms.

- For these, Diamantis-Drewitt proved a Manin period theorem-style algebraicity result for critical L-values of certain weakly holomorphic modular forms.

Further directions

- This gives new structure to explore. It works in other cases; October 2023: Drewitt-Pimm did for Brown's functions.
- What about eigenforms? Guerzhoy gave WHMF eigenforms.
- For these, Diamantis-Drewitt proved a Manin period theorem-style algebraicity result for critical L-values of certain weakly holomorphic modular forms.
- Brown has further conjectures of "motivicness."

Example of producing modular operators using Converse Theorem

- In integral weight, the shadow has a "friend": D^{k-1} the $(k-1)$-fold τ derivative.

Example of producing modular operators using Converse Theorem

- In integral weight, the shadow has a "friend": D^{k-1} the $(k-1)$-fold τ derivative.
- Key intertwining: $D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k\gamma$.

Example of producing modular operators using Converse Theorem

- In integral weight, the shadow has a “friend”: D^{k-1} the $(k-1)$-fold τ derivative.
- Key intertwining: $D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k\gamma$.
- By Converse and *direct* theorems, modualrity-preserving property of D^{k-1} is **equivalent** to an identity on test functions

Example of producing modular operators using Converse Theorem

- In integral weight, the shadow has a "friend": D^{k-1} the $(k-1)$-fold τ derivative.
- Key intertwining: $D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k\gamma$.
- By Converse and *direct* theorems, modualrity-preserving property of D^{k-1} is **equivalent** to an identity on test functions

$$\alpha(\varphi)|_{2-k}W_N = -\alpha(\varphi|_k W_N); \quad \alpha(\varphi) := \mathcal{L}^{-1}(u^{k-1}(\mathcal{L}\varphi)(u))$$

Example of producing modular operators using Converse Theorem

- In integral weight, the shadow has a "friend": D^{k-1} the $(k-1)$-fold τ derivative.
- Key intertwining: $D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k\gamma$.
- By Converse and *direct* theorems, modualrity-preserving property of D^{k-1} is **equivalent** to an identity on test functions

$$\alpha(\varphi)|_{2-k}W_N = -\alpha(\varphi|_k W_N); \quad \alpha(\varphi) := \mathcal{L}^{-1}(u^{k-1}(\mathcal{L}\varphi)(u))$$

- Using special function identities, this boils down to relation for the modified J-Bessel function:

$$J_{-k}(x) = (-1)^k J_k(x).$$

Half-integral weight?

k	Bessel function relation
$\mathbb{Z}$	$J_{-k}(x) = (-1)^k J_k(x)$
$\frac{1}{2} + \mathbb{Z}$	$J_k(x) = \sqrt{\frac{2}{\pi x}} \left(P_k(\frac{1}{x}) \sin x - Q_k(\frac{1}{x}) \cos x\right),$ $J_{-k}(x) = (-1)^{k-\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \left(P_k(\frac{1}{x}) \cos x + Q_k(\frac{1}{x}) \sin x\right),$ for polynomials $P_k(x), Q_k(x)$
$\mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$	No simple relation.

Half-integral weight?

k	Bessel function relation
$\mathbb{Z}$	$J_{-k}(x) = (-1)^k J_k(x)$
$\frac{1}{2} + \mathbb{Z}$	$J_k(x) = \sqrt{\frac{2}{\pi x}} \left(P_k(\frac{1}{x}) \sin x - Q_k(\frac{1}{x}) \cos x\right),$ $J_{-k}(x) = (-1)^{k-\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \left(P_k(\frac{1}{x}) \cos x + Q_k(\frac{1}{x}) \sin x\right),$ for polynomials $P_k(x), Q_k(x)$
$\mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$	No simple relation.

- Branch-Diamantis-Raji-R., 2023: Construct a cohomology class with coefficients in a **finite-dimensional** vector space for half-integral weight cusp forms.

Special L-values

- Before the L-functions, Zagier conjectured that $J(\tau) := j(\tau) - 744$ has "central L-value

$$\text{“}L_J(0)\text{”} = -2\,\mathrm{Re}\left(\int_i^{i+1} J(\tau)\psi(\tau)d\tau\right),$$

where $\psi(s) := \Gamma'(s)/\Gamma(s)$ is the Euler digamma function.

Special L-values

- Before the L-functions, Zagier conjectured that $J(\tau) := j(\tau) - 744$ has "central L-value

$$\text{“}L_J(0)\text{”} = -2\,\mathrm{Re}\left(\int_i^{i+1} J(\tau)\psi(\tau)d\tau\right),$$

where $\psi(s) := \Gamma'(s)/\Gamma(s)$ is the Euler digamma function.

- Bruinier, Funke, and Imamoğlu gave a geometric proof.

Special L-values

- Before the L-functions, Zagier conjectured that $J(\tau) := j(\tau) - 744$ has "central L-value

$$\text{“}L_J(0)\text{”} = -2\,\mathrm{Re}\left(\int_i^{i+1} J(\tau)\psi(\tau)d\tau\right),$$

where $\psi(s) := \Gamma'(s)/\Gamma(s)$ is the Euler digamma function.

- Bruinier, Funke, and Imamoḡlu gave a geometric proof.
- As they point out, this is very similar to formulas for critical L-values of modular forms expressed as cohomological periods of forms over *"spectacle cycles."*

Special L-values

- Before the L-functions, Zagier conjectured that $J(\tau) := j(\tau) - 744$ has "central L-value

$$\text{"}L_J(0)\text{"} = -2\,\mathrm{Re}\left(\int_i^{i+1} J(\tau)\psi(\tau)d\tau\right),$$

where $\psi(s) := \Gamma'(s)/\Gamma(s)$ is the Euler digamma function.
- Bruinier, Funke, and Imamoḡlu gave a geometric proof.
- As they point out, this is very similar to formulas for critical L-values of modular forms expressed as cohomological periods of forms over *"spectacle cycles."*
- Diamantis-Rolen 2022: Full framework for such formulas, using our general L-function technology (Hurwitz+Lerch ζ...)

THANK YOU!!!