TUTORIAL 5

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

This tutorial consists of several practice problems to review some of the main topics from the course.

(1) Find a polynomial $f(x) \in \mathcal{P}_{\leq 2}(\mathbb{R})$ with $f(-1) = 4$, $f(1) = 5$, and $f(-2) = -3$ by following the steps below.
  (a) Interpret the finding of such a polynomial f in terms of solving a certain system of linear equations.
  (b) Write down a matrix equation which is equivalent to this system of linear equations.
  (c) Find the inverse of the matrix you wrote down in (b).
  (d) Use your answer from (c) to solve the system of linear equations, and conclude by writing the polynomial $f(x)$ we are looking for.

Solution:

(a): Suppose that our polynomial is $f(x) = a+bx+cx^2$. Then we are searching for a, b, c for which $f(-1) = a - b + c = 4$, $f(1) = a + b + c = 5$, and $f(-2) = a - 2b + 4c = -3$. That is, we want to solve

$$\begin{cases} a - b + c = 4, \\ a + b + c = 5, \\ a - 2b + 4c = -3. \end{cases}$$

(b): The corresponding matrix equation is $Ax = v$, where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix},$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad v = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}.$$

(c). We can find the inverse A^{-1} in several different ways. Here, we will use the method of cofactors. We first find the matrix of minors, which we recall is the matrix whose (i, j)-th entry is the determinant of the (in this case 2×2) determinant which is obtained by deleting rows i and j of the original matrix.

That is, we have that the matrix of minors is

$$M = \begin{pmatrix} \det\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ \det\begin{pmatrix} -1 & 1 \\ -2 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} & \det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \\ \det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 6 & 3 & -3 \\ -2 & 3 & -1 \\ -2 & 0 & 2 \end{pmatrix}.$$

The corresponding cofactor matrix is obtained by inserting the "checkerboard" pattern of minus and plus signs: (in this case,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

)

$$C = \begin{pmatrix} 6 & -3 & -3 \\ 2 & 3 & 1 \\ -2 & 0 & 2 \end{pmatrix},$$

and the adjugate (or adjoint) is the transpose of this matrix:

$$\begin{pmatrix} 6 & 2 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{pmatrix}.$$

We can also use Laplace expansion along any row or column, using our cofactor matrix C, to find the determinant. Since there is a zero in the last row, we expand along that one:

$$\det A = -2 \cdot 1 + 2 \cdot 4 = -2 + 8 = 6.$$

The inverse of A is then the adjugate we just found above divided by the determinant:

$$A^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

(d): Thus, the solution to our equation $Ax = v$ is

$$x = A^{-1}v = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} \\ \frac{1}{2} \\ -\frac{13}{6}. \end{pmatrix}$$

Thus, the polynomial we are looking for is $f(x) = \frac{20}{3} + \frac{x}{2} - \frac{13x^2}{6}$.

(2) Consider the linear transformation $T\colon \mathcal{P}_{\leq 3}(\mathbb{R}) \to \mathcal{P}_{\leq 2}(\mathbb{R})$ whose action on a polynomial $f(x)$ of degree at most 3 is given by

$$T(f) = f(-1) + f(1)\cdot x + f(-2)\cdot x^2.$$

(a) With respect to the standard bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of these two polynomial spaces, find the matrix representing T.

(b) Using row reduction on this matrix, find bases for the kernel and image of the linear transformation T (recall that these correspond to the kernel and column space of the associated matrix) (Hint: you can make one step easier by using the rank-nullity theorem).

Solution:

(a): We compute the action of T on the basis elements of $\mathcal{P}_{\leq 3}$:

$$T(1) = 1+x+x^2, \quad T(x) = -1+x-2x^2, \quad T(x^2) = 1+x+4x^2, \quad T(x^3) = -1+x-8x^2.$$

These give the columns of the associated matrix, which is thus

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -2 & 4 & -8 \end{pmatrix}.$$

(b): We first find the RREF as follows. First subtract row 1 from row 2 to get

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & -2 & 4 & -8 \end{pmatrix}$$

and then subtract row 1 from row 3 to get

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & -1 & 3 & -7 \end{pmatrix}.$$

Now multiply row 2 by a factor 1/2 to get

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 3 & -7 \end{pmatrix},$$

add row 2 to row 1 to get

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 3 & -7 \end{pmatrix},$$

and add row 2 to row 1 to get

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & -6 \end{pmatrix}.$$

Now multiply row 3 by a factor 1/3 to get

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

and subtract row 3 from row 1 to get the desired RREF

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

To find the kernel of this matrix, we set the free variable x_4 equal to an arbitrary real number t and solve for the pivotal variables x_1, x_2, x_3 to find

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ -t \\ 2t \\ t \end{pmatrix}.$$

This being a generic element of the kernel of A, we find that a basis for the kernel of A is the singleton

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Thus, the kernel of T is one-dimensional and spanned by the polynomial $-2 - x + 2x^2 + x^3$. It is worth checking that this is indeed in the kernel, by plugging in $x = -1, 1, -2$ and checking that all three values give zero.

To find the image of T, we first find the column space of A. We know by the rank-nullity theorem and our computation that the kernel is one-dimensional that the dimension of the column space is $4 - 1 = 3$, and being a subspace of $\mathbb{R}^3$, it must be all of $\mathbb{R}^3$. Thus, a suitable basis is $\{e_1, e_2, e_3\}$. The corresponding basis for the image of T is $\{1, x, x^2\}$.

(3) Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation with $T(1,0,1) = (-1,1,0,2)$, $T(0,1,1) = (0,6,-2,0)$, and $T(-1,1,1) = (4,-2,1,0)$. Find $T(1,2,-1)$. (Hint: Show that $\{(1,0,1),(0,1,1),(-1,1,1)\}$ is a basis of $\mathbb{R}^3$. Now find the coordinate vector of $(1,2,-1)$ with respect to this basis.)

Solution: Consider the set $B = \{(1,0,1),(0,1,1),(-1,1,1)\}$. This is a basis of $\mathbb{R}^3$, as if we consider the matrix with these vectors as columns, we get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

which we find by expanding along the first row has determinant

$$+1 \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-1) \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = 0 + 1 = 1 \neq 0.$$

Now, the change of basis matrix from the standard basis $\{e_1, e_2, e_3'\}$ of $\mathbb{R}^3$ to B is the inverse of the above matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

Using any method (such as the method of cofactors performed on an example below), we find that this inverse is

$$Q = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Thus, the coordinate vector of $(1, 2, -1)$ with respect to this basis is

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -4 \end{pmatrix}.$$

That is,

$$(1, 2, -1) = -3(1, 0, 1) + 6(0, 1, 1) - 4(-1, 1, 1),$$

and so as T is linear we have

$$\begin{aligned} T(1, 2, -1) &= -3T(1, 0, 1) + 6T(0, 1, 1) - 4T(-1, 1, 1) \\ &= -3(-1, 1, 0, 2) + 6(0, 6, -2, 0) - 4(4, -2, 1, 0) \\ &= (-13, 41, 16, -6). \end{aligned}$$

Advanced Problem: In some city, all residents are either at home, at work, or at the cafe. Bored researchers at the municipal office have discovered that if a resident at work at one time, then one hour later, they have a 60% chance of still being at work, a 20% chance of being at home, and a 20% chance of being at the cafe. Similarly, if they are at home, then one hour later they have a 30% chance of begin at work, a 40% chance of still being at home and a 30% chance of being at the cafe. Finally, if they are at the cafe, they have a 10% chance of being at work one hour later, a 10% chance of begin at home, and an 80% of still being in the cafe. After a very long time, about what proportion of residents are at the cafe? (Hint: Given an initial state vector $P = \begin{pmatrix} p_w \\ p_h \\ p_c \end{pmatrix}$, where p_w is the proportion of people at work, p_h is the propotion of people at home, and p_c is the proportion of people at the cafe (with these three numbers adding up to one), after 1 hour, the state vector giving the proportions of people at different places is given by AP where

$$A = \begin{pmatrix} \frac{3}{5} & \frac{3}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{4}{5} \end{pmatrix}.$$

Now find $A^k P$ for large k. This will require a decent amount of calculation, but if you can't solve it during the hour, it is still worthwhile to try to solve it at home or to glance at the solutions and forgetting the arithmetic, follow the argument below.)

Solution:

We follow the hint, and assume the notation there. To find $A^k P$ for large k, we have to diagonalize A. After some algebra, we find that the characteristic polynomial of A is

$$\det(A - \lambda I_3) = \lambda^3 - \frac{9\lambda^2}{5} + \frac{93\lambda}{100} - \frac{13}{100}.$$

We find by inspection that this has a root at $\lambda = 1$, and so we can perform polynomial long division to find that

$$\lambda^3 - \frac{9\lambda^2}{5} + \frac{93\lambda}{100} - \frac{13}{100} = (\lambda - 1)(\lambda^2 - 4\lambda/5 + 13/100).$$

Thus, the eigenvalues are

$$\lambda = 1, \frac{2}{5} \pm \frac{\sqrt{3}}{10}.$$

Since the two latter eigenvalues have absolute value less than 1, they will go away when we take large powers of the corresponding diagonal matrix. That is, we can diagonalize

$$A = SDS^{-1},$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{5} + \frac{\sqrt{3}}{10} & 0 \\ 0 & 0 & \frac{2}{5} - \frac{\sqrt{3}}{10} \end{pmatrix}$$

and where S is the matrix whose columns are the corresponding eigenvectors v_1, v_2, v_3. Thus,

$$A^k = SD^kS^{-1},$$

which for large k is very nearly

$$S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^{-1}.$$

We also find the eigenvectors and plug them in to find that the matrix S is:

$$S = \begin{pmatrix} \frac{1}{2} & 1 - \sqrt{3} & 1 + \sqrt{3} \\ \frac{1}{3} & -2 + \sqrt{3} & -2 - \sqrt{3} \\ 1 & 1 & 1 \end{pmatrix}.$$

We then compute

$$S\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}S^{-1} = \frac{1}{11}\begin{pmatrix}3 & 3 & 3\\ 2 & 2 & 2\\ 6 & 6 & 6\end{pmatrix}.$$

Thus, for very large k, A^k is approximately

$$\frac{1}{11}\begin{pmatrix}3 & 3 & 3\\ 2 & 2 & 2\\ 6 & 6 & 6\end{pmatrix},$$

and A^kP is approximately

$$\begin{pmatrix}\frac{3}{11}(p_c + p_h + p_w)\\ \frac{2}{11}(p_c + p_h + p_w)\\ \frac{6}{11}(p_c + p_h + p_w),\end{pmatrix}$$

which, since $(p_c + p_h + p_w) = 1$, is

$$\begin{pmatrix}\frac{3}{11}\\ \frac{2}{11}\\ \frac{6}{11}\end{pmatrix}.$$

Thus, after a long time, about 6/11, or about 54.5% of residents will be in the cafe.