TUTORIAL 5

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

This tutorial consists of several practice problems to review some of the main topics from the course.

(1) Find a polynomial $f(x) \in \mathcal{P}_{\leq 2}(\mathbb{R})$ with $f(-1) = 4$, $f(1) = 5$, and $f(-2) = -3$ by following the steps below.
   (a) Interpret the finding of such a polynomial f in terms of solving a certain system of linear equations.
   (b) Write down a matrix equation which is equivalent to this system of linear equations.
   (c) Find the inverse of the matrix you wrote down in (b).
   (d) Use your answer from (c) to solve the system of linear equations, and conclude by writing the polynomial $f(x)$ we are looking for.

(2) Consider the linear transformation $T\colon \mathcal{P}_{\leq 3}(\mathbb{R}) \to \mathcal{P}_{\leq 2}(\mathbb{R})$ whose action on a polynomial $f(x)$ of degree at most 3 is given by

$$T(f) = f(-1) + f(1) \cdot x + f(-2) \cdot x^2.$$

   (a) With respect to the standard bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of these two polynomial spaces, find the matrix representing T.
   (b) Using row reduction on this matrix, find bases for the kernel and image of the linear transformation T (recall that these correspond to the kernel and column space of the associated matrix) (Hint: you can make one step easier by using the rank-nullity theorem).

(3) Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation with $T(1,0,1) = (-1,1,0,2)$, $T(0,1,1) = (0,6,-2,0)$, and $T(-1,1,1) = (4,-2,1,0)$. Find $T(1,2,-1)$. (Hint: Show that $\{(1,0,1),(0,1,1),(-1,1,1)\}$ is a basis of $\mathbb{R}^3$. Now find the coordinate vector of $(1,2,-1)$ with respect to this basis.)

Advanced Problem: In some city, all residents are either at home, at work, or at the cafe. Bored researchers at the municipal office have discovered that if a resident at work at one time, then one hour later, they have a 60% chance of still being at work, a 20% chance of being at home, and a 20% chance of being at the cafe. Similarly, if they are at home, then one hour later they have a 30% chance of begin at work, a 40% chance of still being at home and a 30% chance of being at the cafe. Finally, if they are at the cafe, they have a 10% chance of being at work one hour later, a 10% chance of begin at home, and an 80% of still being in the cafe. After a very long time, about what proportion of residents are at the cafe? (Hint: Given an initial state vector $P = \begin{pmatrix} p_w \\ p_h \\ p_c \end{pmatrix}$, where p_w is the proportion of people at work, p_h is the propotion of people at home, and p_c is the proportion of people at the cafe (with these three numbers adding up to one), after 1 hour, the state vector giving the proportions of people at different places is given by AP where

$$A = \begin{pmatrix} \frac{3}{5} & \frac{3}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{4}{5} \end{pmatrix}.$$

Now find $A^k P$ for large k. This will require a decent amount of calculation, but if you can't solve it during the hour, it is still worthwhile to try to solve it at home or to glance at the solutions and forgetting the arithmetic, follow the argument below.)