TUTORIAL 4

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

(1) Let
$$w_1 = (2, 1), \qquad w_2 = (3, 1).$$
Show that $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$. Suppose that T is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ specified by
$$T(w_1) = (1, 5), \qquad T(w_2) = (3, 9),$$
and extended by linearity. Find $T(e_1)$ and $T(e_2)$. Conclude by giving the matrix A for which
$$T(x) = Ax$$
for all $x \in \mathbb{R}^2$.

Solution: The set w_1, w_2 is a basis for $\mathbb{R}^2$ because the determinant of
$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$
is $2 - 3 = -1$, and in particular, non-zero. The values of T at e_1 and e_2 are easily found using linearity once we express e_1 and e_2 in terms of w_1 and w_2. The coordinates of e_1, e_2 are found by multiplying them with the inverse of A, giving
$$A^{-1}e_1 = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} e_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
and
$$A^{-1}e_2 = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} e_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$
That is, we have found that
$$e_1 = -w_1 + w_2, \qquad e_2 = 3w_1 - 2w_2.$$
By linearity, we thus have
$$T(e_2) = -T(w_1) + T(w_2) = (2, 4), \qquad T(e_2) = 3T(w_1) - 2T(w_2) = (-3, -3).$$
The matrix we are looking for is then simply the matrix representing T in the standard basis $E = \{e_1, e_2\}$, i.e., $[T]_E^E$. But this is simply the matrix whose j-th column is $T(e_j)$, and so it is
$$\begin{pmatrix} 2 & -3 \\ 4 & -3 \end{pmatrix}.$$

(2) Find the change of basis matrix from the basis $B_1 = \{1, (x+1), \ldots, (x+1)^n\}$ to the basis $B_2 = \{1, x, \ldots, x^n\}$ for $\mathcal{P}_{\leq n}$ when $n = 1, 2, 3$. How would you find the change of basis matrix going the other direction, namely, from B_2 to B_1?

Solution:

We have to write each of the polynomials in the first basis in terms of the second basis, which is just the standard basis. These are directly read off of the following equations:

$$1 = 1,$$

$$x + 1 = 1 + x,$$

$$(x+1)^2 = 1 + 2x + x^2,$$

$$(x+1)^3 = 1 + 3x + 3x^2 + x^3.$$

Thus, the change of basis matrices from B_1 to B_2 are the matrices whose columns are the coordinate vectors of the elements of B_1 with respect to the standard basis B_2, and hence

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

if $n = 1$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

if $n = 2$, and

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

if $n = 3$. The change of bases matrices going the other direction (from B_2 to B_1) would simply be the inverses of the matrices above.

(3) Consider the linear transformations $S, T \colon \mathbb{R}^2 \to \mathbb{R}^2$, where S is the linear transformation which rotates vectors by $90°$ counter-clockwise and where T is given by

$$T(x, y) = (2x + 3y, 7x - 5y).$$

Compute the matrices associated to these linear transformations (in the standard basis $\{e_1, e_2\}$). Using matrix multiplication, determine the matrix representations for the compositions ST and TS. Two linear transformations are said to **commute** if the compositions are the same in either order, i.e., if $ST = TS$. Using these matrix representations, determine whether S and T commute.

Solution:

The matrix associated to a linear transformation is, in the standard ordered basis, the matrix whose columns are the images of e_1 and e_2 under the transformation. In this case,

$$S(e_1) = e_2 = (0,1) \qquad S(e_2) = -e_1 = (-1,0),$$

so that the matrix associate to S is

$$M_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Similarly,

$$T(e_1) = T(1,0) = (2,7), \qquad T(e_2) = T(0,1) = (3,-5),$$

so that the matrix associate to T is

$$M_T = \begin{pmatrix} 2 & 3 \\ 7 & -5 \end{pmatrix}.$$

The composition ST corresponds to the matrix

$$M_S M_T = \begin{pmatrix} -7 & 5 \\ 2 & 3 \end{pmatrix},$$

while the composition TS corresponds to the matrix

$$M_T M_S = \begin{pmatrix} 3 & -2 \\ -5 & -7 \end{pmatrix}.$$

As these matrices aren't equal, the transformations S and T do not commute.

Advanced problem:

Advanced Problem: Given a finite-dimensional vector space V over $\mathbb{R}$, the **dual space** V^* is the vector space of all **linear functionals**, which are just the linear transformations T mapping $T\colon V \to \mathbb{R}$, under the vector operations of ordinary addition of functions and multiplication of functions by a scalar. Show that the dimension of V^* is the same as the dimension of V. Conclude that V and V^* are isomorphic.

Solution We want to construct a basis for V^* to find its dimension. For this, we need to take a basis of V, say $V = \{v_1, \ldots, v_n\}$. Then we claim that $\{v^{(1)}, \ldots, v^{(n)}\}$ is a basis, where $v^{(j)}$ is the linear transformation uniquely determined (using extension by linearity) by

$$v^{(j)}(e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This set indeed spans V^*, as if $T\colon V \to \mathbb{R}$ is a linear functional, say uniquely determined by $T(v_j) = \alpha_j$, then it is easy to check that $T = \alpha_1 v^{(1)} + \ldots \alpha_n v^{(n)}$, as the action of both of them on a vector $v = c_1 v_1 + \ldots + c_n v_n$ is the real number $c_1\alpha_1 + \ldots + c_n\alpha_n$. Furthermore, the set is linearly independent, as if

$$c_1 v^{(1)} + \ldots + c_n v^{(n)} = 0$$

(note that this is an equality of functions), then this equation says in particular by applying the functions on both sides of the equality to any v_j in the basis that

$$c_j = 0,$$

and so each of the coefficients in the dependency above is 0. Thus, the claimed basis for V^* really is a basis, and it clearly has the same number of elements as the basis for V. As we saw that all finite dimensional vector spaces with the same dimension are isomorphic, we indeed have that $V \cong V^*$.