TUTORIAL 4

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

(1) Let
$$w_1 = (2, 1), \qquad w_2 = (3, 1).$$
Show that $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$. Suppose that T is a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ specified by
$$T(w_1) = (1, 5), \qquad T(w_2) = (3, 9),$$
and extended by linearity. Find $T(e_1)$ and $T(e_2)$. Conclude by giving the matrix A for which
$$T(x) = Ax$$
for all $x \in \mathbb{R}^2$.

(2) Find the change of basis matrix from the basis $B_1 = \{1, (x+1), \dots, (x+1)^n\}$ to the basis $B_2 = \{1, x, \dots, x^n\}$ for $\mathcal{P}_{\leq n}$ when $n = 1, 2, 3$. How would you find the change of basis matrix going the other direction, namely, from B_2 to B_1?

(3) Consider the linear transformations $S, T\colon \mathbb{R}^2 \to \mathbb{R}^2$, where S is the linear transformation which rotates vectors by 90° counter-clockwise and where T is given by
$$T(x, y) = (2x + 3y, 7x - 5y).$$
Compute the matrices associated to these linear transformations (in the standard basis $\{e_1, e_2\}$). Using matrix multiplication, determine the matrix representations for the compositions ST and TS. Two linear transformations are said to **commute** if the compositions are the same in either order, i.e., if $ST = TS$. Using these matrix representations, determine whether S and T commute.

Advanced problem:

Advanced Problem: Given a finite-dimensional vector space V over $\mathbb{R}$, the **dual space** V^* is the vector space of all **linear functionals**, which are just the linear transformations T mapping $T\colon V \to \mathbb{R}$, under the vector operations of ordinary addition of functions and multiplication of functions by a scalar. Show that the dimension of V^* is the same as the dimension of V. Conclude that V and V^* are isomorphic.