TUTORIAL 3

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

(1) The space $\mathcal{P}_{\leq 2}(\mathbb{R})$ of polynomials with real coefficients and degree at most 2 is a vector space over $\mathbb{R}$. Clearly, one basis of it is the set $\{1, x, x^2\}$. Which of the following sets are also bases for $\mathcal{P}_{\leq 2}(\mathbb{R})$?

(a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$

(b) $\{1 + 2x + 4x^2, 3 - 6x^2, x + 3x^2\}$

Solution:

a). Suppose that $\alpha + \beta x + \gamma x^2$ is a generic polynomial in $\mathcal{P}_{\leq 2}(\mathbb{R})$. This polynomial is in $\operatorname{span}(-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2)$ if and only if there are real numbers a, b, c for which

$$a(-1 - x + 2x^2) + b(2 + x - 2x^2) + c(1 - 2x + 4x^2)$$
$$= (-a + 2b + c) + (-a + b - 2c)x + (2a - 2b - 4c)x^2 = \alpha + \beta x + \gamma x^2,$$

or

$$\begin{cases} -a + 2b + c = \alpha \\ -a + b - 2c = \beta \\ 2a - 2b - 4c = \gamma. \end{cases}$$

This system of equations corresponds to the matrix equation

$$\begin{pmatrix} -1 & 2 & 1 \\ -1 & 1 & -2 \\ 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Thus, the set of polynomials spans $\mathcal{P}_{\leq 2}(\mathbb{R})$ if and only if this matrix has column space $\mathbb{R}^3$, which happens if and only if the determinant of the 3×3 matrix of polynomial coefficients is non-zero. Similarly, the set is linearly independent if and only if the only linear combination of the polynomials which is 0 is the trivial with all coefficients zero, which is equivalent to the kernel of the 3×3 matrix being the trivial subspace $\{0\}$, which is also equivalent to the determinant being non-zero.

After a short computation, one finds that

$$\begin{pmatrix} -1 & 2 & 1 \\ -1 & 1 & -2 \\ 2 & -2 & -4 \end{pmatrix} = -8 \neq 0,$$

and so the set of three polynomials is a basis for $\mathcal{P}_{\leq 2}(\mathbb{R})$.

b). Arguing exactly as in a), we form the matrix of polynomials coefficients and take the determinant to find that

$$\det \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 4 & -6 & 3 \end{pmatrix} = 0,$$

and so the polynomials are not a basis. In fact, the third polynomial is $1/2$ times the first minus $1/6$ times the second, and so they are linearly dependent. If you like, you can work out that the first two polynomials are linearly independent, and they span the subspace of polynomials of degree at most two of the form $\alpha + \beta x + \gamma x^2$ with $\gamma = -2\alpha + 3\beta$.

(2) Find a basis of the subspace of skew-symmetric matrices in $M_{3\times 3}(\mathbb{R})$, i.e., those for which $M^T = -M$, as well as a basis for the subspace of symmetric matrices in $M_{3\times 3}(\mathbb{R})$, i.e., those for which $M^T = M$.

For a general matrix space $M_{n\times n}(\mathbb{R})$, show that $M_{n\times n}(\mathbb{R})$ is the direct sum of the subspaces skew-symmetric matrices with the subspace of symmetric matrices. (Hint: The proof is similar to our proof that the space of polynomials is the direct sum of the subspaces of even and odd polynomials.)

Solution:

A general skew symmetric matrix of size 3×3 has the shape

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. Clearly, the set of such matrices is spanned by the set

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Moreover, it is directly clear that these matrices are linearly independent. For example, if a linear combination of them of the shape

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

is the 0 matrix, then clearly $a = b = c = 0$.

The space of symmetric matrices is the set of matrices of the shape

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$, and clearly we can find a basis in a similar manner by setting any of the variables equal to one and the others to be zero, giving the basis

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Following the hint, for a general matrix $A \in M_{n\times n}(\mathbb{R})$, note that if A is both skew-symmetric and symmetric, then $A^T = -A = A$, which implies $A = 0$. Furthermore, A can be written as a sum of a skew-symmetric and a symmetric matrix as follows:

$$A = \frac{A - A^T}{2} + \frac{A + A^T}{2}.$$

Together, these two steps prove the direct sum decomposition claimed.

(3) If $v_1, v_2, v_3 \in V$ form a basis for a vector space V, show that $\{v_1 + v_2 + v_3, v_2 + v_3, v_3\}$ is also a basis.

Solution:

We have to show that the new set of vectors is also linearly independent and spans V. Since v_1, v_2, v_3 span V, to show that our new set of vectors span V, it suffices to show that $v_1, v_2, v_3 \in \operatorname{span}(v_1 + v_2 + v_3, v_2 + v_3, v_3)$. This is verified by checking

$$v_1 = (v_1 + v_2 + v_3) - (v_2 + v_3), \qquad v_2 = (v_2 + v_3) - v_3, \qquad v_3 = v_3.$$

Thus, if $v \in V$, by assumption we can write

$$\begin{aligned} v &= c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1((v_1 + v_2 + v_3) - (v_2 + v_3)) + c_2((v_2 + v_3) - v_3) + c_3 v_3 \\ &= c_1(v_1 + v_2 + v_3) + (c_2 - c_1)(v_2 + v_3) + (c_3 - c_2)v_3 \in \operatorname{span}(v_1 + v_2 + v_3, v_2 + v_3, v_3). \end{aligned}$$

To show that our set of vectors is linearly independent, suppose that

$$c_1(v_1 + v_2 + v_3) + c_2(v_2 + v_3) + c_3 v_3 = c_1 v_1 + (c_1 + c_2)v_2 + (c_1 + c_2 + c_3)v_3 = 0.$$

Since v_1, v_2, v_3 are linearly independent, we have

$$c_1 = 0, \qquad c_1 + c_2 = 0, \qquad c_1 + c_2 + c_3 = 0,$$

which implies that $c_1 = c_2 = c_3 = 0$, implying that our vectors are indeed linearly independent. Hence, they are also a basis for V, as claimed.

Advanced problem:

A university with n students has m societies such that each society has an odd number of members. Any two societies have an even number of common members between them (possibly 0). Show that $m \leq n$. (Hint: Consider each society as a vector in a vector space over the field with two elements $\mathbb{F}_2 = \{0, 1\}$. What do scalar products in this space have to do with shared society membership, where scalar products are defined in F^n for any field F exactly as they are for $\mathbb{R}^n$? Finally, note that in the vector space F^n for any field F, there are at most n linearly independent vectors by the same row reduction argument as we used in $\mathbb{R}^n$.)

Solution: Number the students as $1, 2, \dots n$. Consider the vector space $\mathbb{F}_2^n$. Each society can be represented by a vector $v \in \mathbb{F}_2^n$, where the j-th student is in the society if the j-th component of this vector is 1 and is not in the society if the j-th component of v is 0. We claim that all the society vectors are linearly independent. Indeed, suppose that the societies are $v_1, \dots, v_m$ and that there is a relation

$$v = c_1 v_1 + \dots + c_m v_m = 0$$

where the c_i are all 0 or 1. Consider any society v_j, and take the scalar product

$$0 = v_j \cdot v = \sum_{i=1}^{m} c_i (v_j \cdot v_i).$$

Now each non-zero contribution to the scalar product of two society vectors corresponds to a common member, and so the scalar product $v_j \cdot v_i$ is 1 if the corresponding societies share an odd number of members and 0 if there are an even number of members in both. By the original assumptions, all pairwise scalar products of **different** societies are thus zero, but a society shares an odd number of members with itself by assumption, so $v_j \cdot v_j = 1$. Thus, we have found that

$$0 = \sum_{i=1}^{m} c_i (v_j \cdot v_i) = c_j v_j \cdot v_= c_j.$$

Continuing in this way for each society shows that $c_1, c_2, \dots c_m$ are all 0. Hence, the societies yield a set of linearly independent vectors. As there cannot be more than n linearly independent vectors in $\mathbb{F}_2^n$, we have established the claim.