TUTORIAL 3

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

(1) The space $\mathcal{P}_{\leq 2}(\mathbb{R})$ of polynomials with real coefficients and degree at most 2 is a vector space over $\mathbb{R}$. Clearly, one basis of it is the set $\{1, x, x^2\}$. Which of the following sets are also bases for $\mathcal{P}_{\leq 2}(\mathbb{R})$?
  (a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$
  (b) $\{1 + 2x + 4x^2, 3 - 6x^2, x + 3x^2\}$

(2) Find a basis of the subspace of skew-symmetric matrices in $M_{3\times 3}(\mathbb{R})$, i.e., those for which $M^T = -M$, as well as a basis for the subspace of symmetric matrices in $M_{3\times 3}(\mathbb{R})$, i.e., those for which $M^T = M$.

For a general matrix space $M_{n\times n}(\mathbb{R})$, show that $M_{n\times n}(\mathbb{R})$ is the direct sum of the subspaces skew-symmetric matrices with the subspace of symmetric matrices. (Hint: The proof is similar to our proof that the space of polynomials is the direct sum of the subspaces of even and odd polynomials.)

(3) If $v_1, v_2, v_3 \in V$ form a basis for a vector space V, show that $\{v_1 + v_2 + v_3, v_2 + v_3, v_3\}$ is also a basis.

Advanced problem:

A university with n students has m societies such that each society has an odd number of members. Any two societies have an even number of common members between them (possibly 0). Show that $m \leq n$. (Hint: Consider each society as a vector in a vector space over the field with two elements $\mathbb{F}_2 = \{0, 1\}$. What do scalar products in this space have to do with shared society membership, where scalar products are defined in F^n for any field F exactly as they are for $\mathbb{R}^n$? Finally, note that in the vector space F^n for any field F, there are at most n linearly independent vectors by the same row reduction argument as we used in $\mathbb{R}^n$.)