HOMEWORK 8

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, December 1**. Please write your name and course on your assignment, and make sure to staple your papers.

(1) Recalling that $\mathcal{P}_{\leq 2}$ is the space of polynomials of degree at most 2 with real coefficients, consider the linear transformation $T\colon \mathcal{P}_{\leq 2} \to \mathbb{R}^2$ given by

$$T(P(x)) = (P(0), P(1)).$$

For example, we have $T(x^2 + 1) = (1, 2)$.

(a) Find the matrix associated to T in terms of the standard bases $\{1, x, x^2\}$ of $\mathcal{P}_{\leq 2}$ and $\{e_1, e_2\}$ of $\mathbb{R}^2$.

(b) Find a basis for the kernel of the matrix you determined in part a). The elements of this basis will be coordinate vectors of a basis for the kernel of the linear transformation T. Write this basis for the kernel of T.

Solution: a); The columns of this matrix are determined by taking the vectors in the first basis, applying the linear transformation, and then computing the coordinate vectors of these images (which won't require a computation in this case as we are using the standard basis of $\mathbb{R}^2$). We thus compute $T(1) = (1, 1)$, $T(x) = (0, 1)$, $T(x^2) = (0, 1)$. Thus, the associated matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

b): Note that the RREF of A is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The kernel is found by setting the free parameter $x_3 = t$ and solving for the pivotal variables x_1, x_2, and so we get the set of vectors in the kernel are those of the form

$$\begin{pmatrix} 0 \\ -t \\ t \end{pmatrix}.$$

A basis for this kernel is $(0, -1, 1)$. The corresponding polynomial with this coordinate vector is $x^2 - x$. Thus, a basis for the kernel of T is given by $\{x^2 - x\}$. There is another way to see this, as if we require that $P(0) = P(1) = 0$, then by properties of polynomials we also know that $P(x) = x(x-1)Q(x)$ for some other polynomial Q. Since the degree of $x(x-1)$ is already 2, in our situation, Q has to be a degree zero polynomial, i.e., a constant, if the degree of P is at most 2.

(2) Find the matrix representation of the linear map T in Problem 1 in terms of the non-standard bases $\{1-x, x, -2+x^2\}$ of $\mathcal{P}_{\leq 2}$ and $\{(1,2),(3,4)\}$ of $\mathbb{R}^2$.

Solution:

We first compute the images of elements of the first basis with respect to T:

$$T(1-x) = (1,0), \qquad T(x) = (0,1), \qquad T(-2+x^2) = (-2,-1).$$

Now, we must find the coordinate vectors of these three vectors in terms of our non-standard basis of $\mathbb{R}^2$. Recall that we do this by forming the matrix whose columns are our basis elements:

$$C = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},$$

taking the inverse

$$C^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix},$$

and multiplying with the vectors we found above:

$$C^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad C^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}, \qquad C^{-1}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -3/2 \end{pmatrix}.$$

These are then the coordinate vectors of our images, and give the columns of the matrix we are looking for:

$$\begin{pmatrix} -2 & \frac{3}{2} & \frac{5}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

(3) Consider the linear transformation $f\colon M_{2\times 2} \to M_{2\times 2}$ given by

$$f(A) = A + A^T,$$

where A^T is the transpose of A.

(a) Consider the standard basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_{2\times 2}$. Find the matrix representing f with respect to this basis.

(b) Find a basis for the kernel of the matrix you found in part a). The set of matrices with the vectors in this basis being their coordinate vectors is a basis for the kernel of f. Write down this basis.

Solution:

a): We compute the image under f of the basis vectors:

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$f\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$f\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}.$$

The coordinate vectors of these images, respectively, are $(2,0,0,0), (0,1,1,0), (0,1,1,0), (0,0,0,2)$. The corresponding matrix has these vectors as its columns, and is hence given by

$$A = \begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

b): The RREF of A is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

The variable x_3 is a free parameter which we set equal to t, and after solving for the pivotal variables x_1, x_2, x_4, we find that the kernel of A is the span of the vector $(0,-1,1,0)$. This is the coordinate vector for the matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. The set consisting of this matrix is thus a basis for the kernel of f. Note that we are really computing a basis for the set of matrices with $A + A^T = 0$, or $A^T = -A$, so the skew symmetric matrices, for which we have computed a basis in the 3×3 case.

(4) Suppose that $g(x) = 3 + x$. Consider the linear transformation $T\colon \mathcal{P}_{\le 2} \to \mathcal{P}_{\le 2}$ given by
$$T(P(x)) = P'(x)\cdot g(x) + 2P(x),$$
were $'$ denotes the derivative d/dx. Now consider the linear transformation $U\colon \mathcal{P}_{\le 2} \to \mathbb{R}^3$ given by
$$U(a + bx + cx^2) = (a+b, c, a-b).$$
  (a) In terms of the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_{\le 2}$, compute the matrix representing T.
  (b) In terms of the standard basis of $\mathcal{P}_{\le 2}$ and the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, compute the matrix representing U.
  (c) Using the matrices you found in a) and b), find the matrix representing the composition UT by taking the product of these two matrices.

(d) Directly compute the matrix representing the composition UT using the same bases as above, and check that the result is the same as the matrix product you took in part c).

Solution:

a): The images of the basis vectors for $\mathcal{P}_{\leq 2}$ are $T(1) = 0 + 2 = 2$, $T(x) = g(x) + 2x = 3 + 3x$, $T(x^2) = 2xg(x) + 2x^2 = 6x + 2x^2 + 2x^2 = 6x + 4x^2$. The corresponding matrix can be read off of these three equations as

$$A_T = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

b). The images of the basis elements in $\mathcal{P}_{\leq 2}$ under U are $U(1) = (1, 0, 1)$, $U(x) = (1, 0, -1)$, and $U(x^2) = (0, 1, 0)$. Hence, the corresponding matrix is

$$A_U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

c). The product of these two matrices is

$$A_U A_T = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

d). The images of the basis elements of $\mathcal{P}_{\leq 2}$ under UT are directly computed, continuing the computation in part a), to be $UT(1) = U(2) = (2, 0, 2)$, $UT(x) = U(3 + 3x) = (6, 0, 0)$, and $UT(x^2) = U(6x + 4x^2) = (6, 4, -6)$. These are indeed the columns of the matrix $A_U A_T$ we computed in c), and so the matrix corresponding to the composition UT is indeed that computed in c) using the matrix product.