HOMEWORK 5

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, November 3**. Please put your name and course on your assignment, and make sure to staple your papers.

(1) For each of the following examples, determine whether it is or isn't a vector space. If it isn't, show why it doesn't satisfy at least one of the axioms for a vector space, and otherwise prove that its a vector space.

(a) The set of polynomials of degree n with real coefficients, where n is a positive integer.

(b) The set of *skew-symmetric matrices* of size $n \times n$ with real entries, i.e., those matrices M for which $M^T = -M$.

(c) The set of points $\{(x, y) \in \mathbb{R}^2 : y = x^3 - x.\}$.

Solution:

a): This is not a vector space, as it doesn't contain the polynomial 0 (since this has degree zero). Specifically, any vector space must contain a vector 0 with $0 + v = v$ for all $v \in V$, and clearly the only polynomial that could satisfy this is the zero polynomial, which isn't included. This can be fixed if you instead consider the set of polynomials of degree **at most** n, which is a vector space.

b): This is a vector space. As its a subset of $M_{n\times n}(\mathbb{R})$, we just have to check that it satisfies our subspace criterion theorem conditions. Firstly, clearly 0 is skew-symmetric, as $0^T = 0 = -0$. Now suppose that A and B are skew symmetric, and that $c \in \mathbb{R}$. By the basic properties of transposes which we gave in a previous lecture,

$$(A + B)^T = A^T + B^T = -A - B = -(A + B),$$

$$(cA)^T = c(A^T) = c(-A) = -(cA),$$

so that $A + B$ and cA are skew-symmetric. By our main theorem on subspaces, this is a subspace of $M_{n\times n}(\mathbb{R})$, and hence is a vector space itself.

c). This is not a vector space. For example, the point $(1, 0)$ is in the set, but $(2, 0)$ is not, as $2^3 - 2 = 6 \neq 0$. Hence, the set isn't closed under scalar multiplication or vector addition.

(2) The set $\mathbb{R}^+$ of positive real numbers is not a vector space over the field $\mathbb{R}$ with the usual operations of addition and multiplication of real numbers. For example, it isn't closed under scalar multiplication as, eg., $1 \in \mathbb{R}^+$ but $-1 \cdot 1 = -1$ is not in $\mathbb{R}^+$. However, we can make the set $\mathbb{R}^+$ be a vector space over $\mathbb{R}$ if we use different operations for vector addition and scalar multiplication. To distinguish

them from ordinary multiplication and addition, we will write them as $\otimes$ and $\oplus$, and we define them by

$$c \otimes v = v^c$$

(i.e., scalar multiplication of a real number c and a positive real number v is the number v^c) and

$$v \oplus w = vw$$

(i.e., the vector addition of two positive real numbers v and w is the product vw).

Show that these two operations do indeed turn $\mathbb{R}^+$ into a vector space by checking the 10 parts of the definition of a vector space we gave in class.

Solution: We check the axioms in the definition of vector spaces one at a time, following the order and notation from class.

(1): $\mathbb{R}^+$ is closed under vector addition as the product of two positive real numbers (in the usual sense) is a positive real number.

(2): The set $\mathbb{R}^+$ is closed under scalar multiplication as if $c, v \in \mathbb{R}$ with $v > 0$, then $v^c > 0$ too.

(3): The commutativity claim states that $vw = wv$ for real numbers v, w, which is true.

(4): This property becomes $(uv)w = u(vw)$ for real numbers u, v, w; i.e., this is equivalent to associativity of multiplication in $\mathbb{R}$.

(5): The vector 0 we are looking for is the real number $1 \in \mathbb{R}^+$, since $1 \oplus v = 1 \cdot v = v$ for all $v \in \mathbb{R}^+$.

(6): The additive inverse of $v \in \mathbb{R}^+$ is $1/v \in \mathbb{R}^+$.

(7): Associativity of scalar multiplication says that $c \otimes (c' \otimes v) = (c \cdot c') \otimes v$. Rewriting, this is $(v^{c'})^c = v^{c \cdot c'}$, which is a standard rule for exponentiation of real numbers.

(8): This property becomes $c \otimes (v \oplus w) = (c \otimes v) \oplus (c \otimes w)$, or, in other words, $(vw)^c = (v^c) \cdot (w^c)$, which is true.

(9): This condition states that $(c + c') \otimes v = c \otimes v \oplus c' \otimes v$, or $v^{c+c'} = v^c \cdot v^{c'}$, which is true.

(10): The multiplicative identity for $\mathbb{R}$ is 1, and indeed we have $c \otimes v = v^1 = v$ for all $v \in \mathbb{R}^+$.

(3) In HW 2, you showed that the span of the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

is the plane $y = 3x$ in $\mathbb{R}^3$. Another way of saying this is that the column space, $\mathrm{col}(A)$ of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

is the plane $y = 3x$. Of course, since the origin is on this plane, the plane is subspace of $\mathbb{R}^3$. In this question, you will continue working with this example.

(a) Find the kernel of the transpose, $\ker(A^T)$, using row reduction. What type of geometric object is it?

(b) Show that we have the direct sum decomposition

$$\mathbb{R}^3 = \mathrm{col}(A) \oplus \ker(A^T).$$

Solution:

a): To find the kernel, we row reduce the system corresponding to the equation $A^T x = 0$, giving

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right),$$

which has RREF

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This equation has a solution set which is the line $L = \{(x, y, z) : x + 3y = 0, z = 0\}$, or equivalently it is the line $\{tv : t \in \mathbb{R}\}$ where $v = (-3, 1, 0)$.

b): To show that $\mathbb{R}^3 = \mathrm{col}(A) \oplus \ker(A^T)$, by definition we must show

$$\mathrm{col}(A) \cap \ker(A^T) = \{0\},$$

and

$$\mathbb{R}^3 = \mathrm{col}(A) + \ker(A).$$

The first equation follows easily, since if a point (x, y, z) is on both the line and the plane, then $x = z = 0$ and $y = -x/3 = -0/3 = 0$, so that $(x, y, z) = 0$. For the second condition, suppose that $(x, y, z) \in \mathbb{R}^3$. Then we want to show that we write it as a sum of a general vector on the plane, say $(a, 3a, b)$ and a vector on the line, say $(-3c, c, 0)$. That is, we want to show that there are real numbers a, b, c for which

$$(x, y, z) = (a, 3a, b) + (-3c, c, 0) = (a - 3c, 3a + c, b),$$

or $x = a - 3c$, $y = 3a + c$, and $b = z$. The system of the first two equations is easily solved, yielding

$$a = \frac{x}{10} + \frac{3y}{10}, c = -\frac{3x}{10} + \frac{y}{10}.$$

Hence, the arbitrary point (x, y, z) can be written as

$$(x, y, z) = \left(\frac{x}{10} + \frac{3y}{10}, \frac{3x}{10} + \frac{9y}{10}, z\right) + \left(\frac{9x}{10} - \frac{3y}{10}, -\frac{3x}{10} + \frac{y}{10}, 0\right),$$

where the first vector on the right hand side is on the plane and the second vector is on the line.

Notice that something very interesting happened here. More precisely, note that the line, which was the kernel of the transpose, is actually a normal vector for the plane which was the column space. We have just performed a manual check of an interesting, and much more general important result.

(4) Let W be a subset of V, a vector space over a field F. Show that W is a subspace if and only if $0 \in W$ and $cw_1 + w_2 \in W$ for all $c \in F$ and $w_1, w_2 \in W$ (hint: use the theorem we stated in class which gave criteria to check when a subset is a vector space).

Solution:

We show that these conditions are satisfied if and only if the three conditions in our theorem on subspaces are satisfied, i.e., if and only if $0 \in W$, W is closed under addition of vectors, and W is closed under scalar multiplication by elements of F.

Suppose firstly that these three conditions hold. Then we must show that this implies $0 \in W$ and $cw_1 + w_2 \in W$ for all $c \in F$ and $w_1, w_2 \in W$. The condition $0 \in W$ was already assumed, and the second condition follows since the third condition of our theorem implies that if $cw_1 \in W$ and the second one then implies that $cw_1 + w_2 \in W$, as desired.

Conversely, suppose that $0 \in W$ and $cw_1 + w_2 \in W$ for all $c \in F$ and $w_1, w_2 \in W$. Then we must show that the three conditions of our theorem hold. The first, that $0 \in W$, is trivial. The second, that W is closed under vector addition, follows by taking $c = 1$. The third, that $cw_1 \in W$ for all $c \in F$, $w_1 \in W$, follows by taking $w_2 = 0$.

Note that this problem sometimes gives a more convenient set of conditions to check, as we can combine conditions 2 and 3 of our subspace theorem into one condition, so that we only have two conditions to check instead of three.