HOMEWORK 5

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, November 3**. Please put your name and course on your assignment, and make sure to staple your papers.

(1) For each of the following examples, determine whether it is or isn't a vector space. If it isn't, show why it doesn't satisfy at least one of the axioms for a vector space, and otherwise prove that its a vector space.
   (a) The set of polynomials of degree n with real coefficients, where n is a positive integer.
   (b) The set of *skew-symmetric matrices* of size $n \times n$ with real entries, i.e., those matrices M for which $M^T = -M$.
   (c) The set of points $\{(x, y) \in \mathbb{R}^2 : y = x^3 - x.\}$.

(2) The set $\mathbb{R}^+$ of positive real numbers is not a vector space over the field $\mathbb{R}$ with the usual operations of addition and multiplication of real numbers. For example, it isn't closed under scalar multiplication as, eg., $1 \in \mathbb{R}^+$ but $-1 \cdot 1 = -1$ is not in $\mathbb{R}^+$. However, we can make the set $\mathbb{R}^+$ be a vector space over $\mathbb{R}$ if we use different operations for vector addition and scalar multiplication. To distinguish them from ordinary multiplication and addition, we will write them as $\otimes$ and $\oplus$, and we define them by

$$c \otimes v = v^c$$

(i.e., scalar multiplication of a real number c and a positive real number v is the number v^c) and

$$v \oplus w = vw$$

(i.e., the vector addition of two positive real numbers v and w is the product vw).

Show that these two operations do indeed turn $\mathbb{R}^+$ into a vector space by checking the 10 parts of the definition of a vector space we gave in class.

(3) In HW 2, you showed that the span of the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

is the plane $y = 3x$ in $\mathbb{R}^3$. Another way of saying this is that the column space, $\mathrm{col}(A)$ of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

is the plane $y = 3x$. Of course, since the origin is on this plane, the plane is subspace of $\mathbb{R}^3$. In this question, you will continue working with this example.

(a) Find the kernel of the transpose, $\ker(A^T)$, using row reduction. What type of geometric object is it?

(b) Show that we have the direct sum decomposition

$$\mathbb{R}^3 = \mathrm{col}(A) \oplus \ker(A^T).$$

(4) Let W be a subset of V, a vector space over a field F. Show that W is a subspace if and only if $0 \in W$ and $cw_1 + w_2 \in W$ for all $c \in F$ and $w_1, w_2 \in W$ (hint: use the theorem we stated in class which gave criteria to check when a subset is a vector space).