HOMEWORK 3

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, October 20**. Please put your name and student number, as well as your subject (Maths., TP, or TSM) on the back of your assignment, and make sure to staple your papers.

(1) Suppose that

$$A = \begin{pmatrix} 4 & 0 \\ 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \end{pmatrix}.$$

(a) Compute AB and BA.
(b) Compute, or show that it doesn't exist, the following inverses: $(AB)^{-1}$ and $(BA)^{-1}$.

Solution:
(a):
We find that

$$AB = \begin{pmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \\ 1 & 0 & -3 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 2 & -2 \\ -4 & -3 \end{pmatrix}.$$

(b):
The first matrix AB isn't invertible. For example, note that the first row is twice the second, and so we can row-reduce AB to have a row all zeros. Hence, its RREF isn't I_3, and it isn't invertible.

The second matrix is invertible. We compute it by row reducing the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -4 & -3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ -4 & -3 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & -7 & 2 & 1 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} \end{array}\right),$$

and so $(BA)^{-1} = \begin{pmatrix} \frac{3}{14} & -\frac{1}{7} \\ -\frac{2}{7} & -\frac{1}{7} \end{pmatrix}$.

(2) For any 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the trace of A, denoted $\mathrm{tr}(A)$, as the sum of diagonal elements, i.e. $\mathrm{tr}(A) = a + d$. Further let A^2 be the product $A \cdot A$ for any square matrix A. Show that for any 2×2 matrix, we have

$$2 \det A = \det \begin{pmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{pmatrix}.$$

Hint: you may use the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Solution:
We first compute

$$\mathrm{tr}(A) = a + d,$$

$$\mathrm{tr}(A^2) = \mathrm{tr}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \mathrm{tr} \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = a^2 + 2bc + d^2.$$

Hence,

$$\det \begin{pmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{pmatrix} = \det \begin{pmatrix} a+d & 1 \\ a^2+2bc+d^2 & a+d \end{pmatrix}$$
$$= (a+d)^2 - (a^2+2bc+d^2) = 2ad - 2bc = 2 \det A.$$

(3) Consider the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 7 & 5 & 6 \end{pmatrix}.$$

(a) Compute $\pi\sigma$.
(b) Write $\pi\sigma$ as a product of disjoint cycles.
(c) Use your answer from the last part to write $\pi\sigma$ as a product of transpositions.
(d) Use you answer from the last part to find $\mathrm{sign}(\pi\sigma)$.

Solution:
(a):
We find that

$$\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 1 & 3 & 6 & 5 & 2 \end{pmatrix}.$$

(b): We compute $\pi\sigma = (17243)(56)$.
(c): We find that $\pi\sigma = (13)(14)(12)(17)(56)$.
(d): Since there were 5 transpositions in the representation for $\pi\sigma$ we found in (c), $\pi\sigma$ is an odd permutation, i.e., $\mathrm{sign}(\pi\sigma) = -1$.

(4) Compute $\det A$ **directly from the definition** (that is, as the unique alternating multilinear function on rows which has $\det I_n = 1$), where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 9 \end{pmatrix}.$$

Solution:

Writing this as a function of rows, and using multilinearity and the alternating property of det, we find that

$$\det A = \det(e_1 + 2e_2 + 3e_3, 5e_2, 7e_1 + 9e_3)$$
$$= 5\det(e_1, e_2, 7e_1 + 9e_3) + 10\det(e_2, e_2, 7e_1 + 9e_3) + 15\det(e_3, e_2, 7e_1 + 9e_3)$$
$$= 35\det(e_1, e_2, e_1) + 45\det(e_1, e_2, e_3) + 70\det(e_2, e_2, e_1) + 90\det(e_2, e_2, e_3)$$
$$+ 105\det(e_3, e_2, e_1) + 135\det(e_3, e_2, e_3)$$
$$= 45\det(e_1, e_2, e_3) + 105\det(e_3, e_2, e_1) = 45\det(e_1, e_2, e_3) - 105\det(e_1, e_2, e_3)$$
$$= 45\det(I_3) - 105\det(I_3) = 45 - 105 = -60.$$