HOMEWORK 2 SOLUTIONS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

Solutions are due at the beginning of class on **Thursday, October 13**. Please put your name and student number, as well as your subject (Maths., TP, or TSM) on the back of your assignment, and make sure to staple your papers. In general, work must always be shown to get full credit.

(1) Use Gauss-Jordan elimination to solve the system of linear equations

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 12 \\ 2x_1 + 4x_2 + 12x_3 = -17 \\ x_1 - 4x_2 - 12x_3 = 22. \end{cases}$$

Solution: This system corresponds to the matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{array}\right).$$

Row reducing, we get

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{array}\right) \xrightarrow{(2)\mapsto(2)-2(1),\ (3)\mapsto(3)-(1)} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 8 & 24 & -41 \\ 0 & -2 & -6 & 10 \end{array}\right)$$

$$\xrightarrow{(2)\leftrightarrow(3)} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 8 & 24 & -41 \end{array}\right) \xrightarrow{(3)\mapsto(3)+4(2)} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{array}\right).$$

From the last line, we see that the system is inconsistent, and hence that there are no solutions.

(2) Consider the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 4 & 9 \end{array}\right).$$

(a) Write the system of linear equations corresponding to this augmented matrix.
(b) Find the reduced row echelon form of this matrix.
(c) Use your answer from (b) to determine a solution set for the system you found in (a).

Solution:

(a): In the variables x_1, x_2, x_3, x_4, we can write the system of linear equations corresponding to the matrix as

$$\begin{cases} x_1 - 2x_2 + 2x_3 - x_4 = 3 \\ 3x_1 + x_2 + 6x_3 + 11x_4 = 16 \\ 2x_1 - x_2 + 4x_3 + 4x_4 = 9. \end{cases}$$

(b): We reduce as follows:

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 4 & 9 \end{array}\right) \xrightarrow{(2)\mapsto(2)-3(1)\ \ (3)\mapsto(3)-2(1)} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 7 & 0 & 14 & 7 \\ 0 & 3 & 0 & 6 & 3 \end{array}\right)$$
$$\xrightarrow{(3)\mapsto(3)-\frac{3}{7}(2)} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 7 & 0 & 14 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{(2)\mapsto\frac{1}{7}(2)} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{(1)+2(2)} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

(c). There are pivots in the first and second column, and there are none in the third or fourth columns. Thus, x_1, x_2 are pivotal variables and x_3, x_4 are free variables. Thus, we set $x_3 = t_3$, $x_4 = t_4$ where t_3, t_4 are real parameters, and solve for the first two variables to obtain the solution set

$$\begin{cases} x_1 = 5 - 2t_3 - 3t_4 \\ x_2 = 1 - 2t_4 \\ x_3 = t_3 \\ x_4 = t_4 \end{cases}.$$

(3) Find the span in $\mathbb{R}^3$ of the vectors

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Solution: We want to find all vectors $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $c_1v_1+c_2v_2+c_3v_3 = v$ for constants $c_1, c_2, c_3 \in \mathbb{R}$. That is, we want to consider x, y, z fixed and find the solution to the system

$$\begin{cases} c_1 + c_3 = x \\ 3c_1 + 3c_3 = y \\ 3c_1 + c_2 + c_3 = z, \end{cases}$$

which corresponds to the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 3 & 0 & 3 & y \\ 3 & 1 & 1 & z \end{array}\right).$$

We row-reduce as follows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 3 & 0 & 3 & y \\ 3 & 1 & 1 & z \end{array}\right) \xrightarrow{(2)\mapsto(2)-3(1),\ (3)\mapsto(3)-3(1)} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 0 & 0 & y-3x \\ 0 & 1 & -2 & z-3x \end{array}\right)$$
$$\xrightarrow{(2)\leftrightarrow(3)} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & -2 & z-3x \\ 0 & 0 & 0 & y-3x \end{array}\right).$$

This can only have a solution if $y = 3x$, as otherwise the system is inconsistent. As long as this is satisfied, however, the system will be consistent and will have one free parameter, and in particular will have a solution. Hence, the span of the three vectors is the plane $y = 3x$.

(4) Find the reduced row echelon form of the matrix

$$\begin{pmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Solution: We reduce as follows:

$$\begin{pmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{(1)\mapsto\frac{1}{2}(1)} \begin{pmatrix} 1 & 2 & 1 & \frac{1}{2} \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{(2)\mapsto(2)-4(1)\ \ (3)\mapsto(3)+6(1)} \begin{pmatrix} 1 & 2 & 1 & \frac{1}{2} \\ 0 & -5 & -4 & -3 \\ 0 & 12 & 8 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{(2)\mapsto-\frac{1}{5}(2)} \begin{pmatrix} 1 & 2 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \\ 0 & 12 & 8 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{(1)\mapsto(1)-2(2)\ \ (3)\mapsto(3)-12(2)\ \ (4)\mapsto(4)-(2)} \begin{pmatrix} 1 & 0 & -\frac{3}{5} & -\frac{7}{10} \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \\ 0 & 0 & -\frac{8}{5} & -\frac{21}{5} \\ 0 & 0 & \frac{1}{5} & \frac{7}{5} \end{pmatrix},$$

$$\xrightarrow{(3)\mapsto-\frac{5}{8}(3)} \begin{pmatrix} 1 & 0 & -\frac{3}{5} & -\frac{7}{10} \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{21}{8} \\ 0 & 0 & \frac{1}{5} & \frac{7}{5} \end{pmatrix} \xrightarrow{(1)\mapsto(1)+\frac{3}{5}(3),(2)\mapsto(2)-\frac{4}{5}(3),(4)\mapsto(4)-\frac{1}{5}(3)} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{8} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{21}{8} \\ 0 & 0 & 0 & \frac{7}{8} \end{pmatrix}$$

$$\xrightarrow{(4)\mapsto\frac{8}{7}(4)} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{8} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{21}{8} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1)\mapsto(1)-\frac{7}{8}(4),(2)\mapsto(2)+\frac{3}{2}(4),(3)\mapsto(3)+\frac{21}{8}(4)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is our desired reduced row echelon form.