HOMEWORK 1 SOLUTIONS

MA1111: LINEAR ALGEBRA I, MICHAELMAS 2016

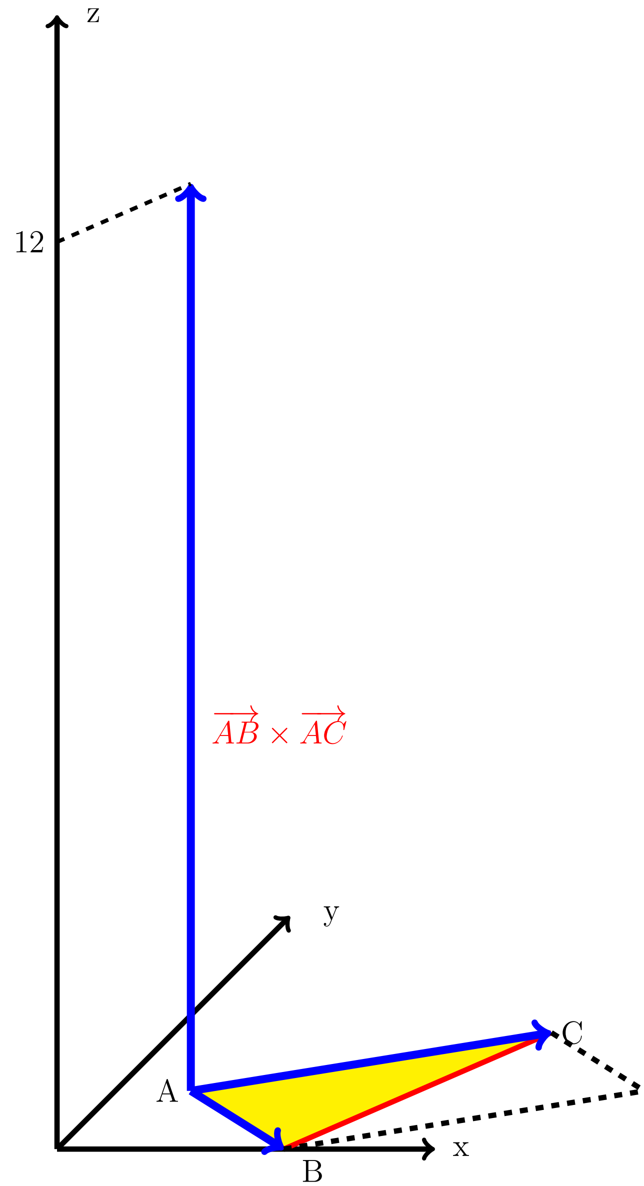
Solutions are due at the beginning of class on **Thursday, October 6**. Also attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism. Please put your name and student number, as well as your subject (Maths., TP, or TSM) on the back of your assignment, and make sure to staple your papers. In general, work must always be shown to get full credit.

(1) Find the area of the triangle in the plane with vertices specified by their $3D$-coordinates $A = (1, 2, 0)$, $B = (3, 0, 0)$, and $C = (5, 4, 0)$. (Hint: this triangle constitutes half of a parallelogram.) You must use a result we learned from class, and cannot use assume other geometric results such as Heron's formula!

Solution: We first find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Recall that these are obtained algebraically by subtracting coordinates, yielding

$$\overrightarrow{AB} = (2, -2, 0), \qquad \overrightarrow{AC} = (4, 2, 0).$$

We then compute their cross product to be $\overrightarrow{AB} \times \overrightarrow{AC} = (0, 0, 12)$, which of course has length 12. The area of this triangle is half the area of this parallelogram, namely 6. Here is an accompanying diagram of the situation:



(2) Show (i.e., prove) that if v, w, w' are any three-dimensional vectors, then

$$v \times (w + w') = v \times w + v \times w'.$$

Solution: We give the components of each vector names, calling

$$v = (v_1, v_2, v_3), \qquad w = (w_1, w_2, w_3), \qquad w' = (w'_1, w'_2, w'_3).$$

We can show the equality by computing the difference between the left-hand side and right-hand-sides to be zero:

$$v \times (w + w') - v \times w - v \times w' = (v_1, v_2, v_3) \times ((w_1 + w_1', w_2 + w_2', w_3 + w_3')) - v \times w + v \times w'$$
$$= \Big(v_2(w_3 + w_3') - v_3(w_2 + w_2')), v_3(w_1 + w_1') - v_1(w_3 + w_3'), v_1(w_2 + w_2') - v_2(w_1 + w_1') \Big)$$
$$- \Big(v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \Big) - \Big(v_2 w_3' - v_3 w_2', v_3 w_1' - v_1 w_3', v_1 w_2' - v_2 w_1' \Big).$$

This is easily seen by inspection to equal 0.

(3) In 4-dimensional (Euclidean) space $\mathbb{R}^4$, consider the "standard basis vectors" (these are the analogues of i, j, k in three-dimensional space) defined by

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1).$$

Suppose that f is a **multilinear** function $f(x, y, z)$ taking inputs of 4-d vectors x, y, z and outputting a real number. Suppose that the value of f on the standard basis vectors is determined by the equation

$$f(e_i, e_j, e_k) = i \cdot j - k$$

for $i, j, k = 1, 2, 3, 4$. For example, $f(e_2, e_3, e_4) = 2 \cdot 3 - 4 = 2$. Then find the value $f((8, 1, 3, 2), (5, 0, 7, 0), (0, 0, 0, 1))$.

Solution: We first write the inputs in terms of the unit basis vectors. This gives

$$f((8, 1, 3, 2), (5, 0, 7, 0), (0, 0, 0, 1)) = f(8e_1 + e_2 + 3e_3 + 2e_4, 5e_1 + 7e_3, e_4).$$

Using multilinearity, we decompose this as

$$f(8e_1 + e_2 + 3e_3 + 2e_4, 5e_1 + 7e_3, e_4)$$
$$= 8f(e_1, 5e_1 + 7e_3, e_4) + f(e_2, 5e_1 + 7e_3, e_4) + 3f(e_3, 5e_1 + 7e_3, e_4) + 2f(e_4, 5e_1 + 7e_3, e_4)$$
$$= 40f(e_1, e_1, e_4) + 56f(e_1, e_3, e_4) + 5f(e_2, e_1, e_4) + 7f(e_2, e_3, e_4)$$
$$+ 15f(e_3, e_1, e_4) + 21f(e_3, e_3, e_4) + 10f(e_4, e_1, e_4) + 14f(e_4, e_3, e_4).$$

Using the definition for f in terms of its values on unit basis vectors, we find

$$f(8e_1 + e_2 + 3e_3 + 2e_4, 5e_1 + 7e_3, e_4)$$
$$= 40(1 \cdot 1 - 4) + 56(1 \cdot 3 - 4) + 5(2 \cdot 1 - 4) + 7(2 \cdot 3 - 4)$$
$$+ 15(3 \cdot 1 - 4) + 21(3 \cdot 3 - 4) + 10(4 \cdot 1 - 4) + 14(4 \cdot 3 - 4)$$
$$= 30.$$

(4) (a). Find the equation of the plane in $\mathbb{R}^3$ passing through the points $A = (1, 1, 6)$, $B = (3, 4, 2)$, and $C = (5, 4, 1)$.

Solution: We first compute

$$\overrightarrow{AB} = (2, 3, -4),$$

and
$$\overrightarrow{AC} = (4, 3, -5).$$
Then we compute $\overrightarrow{AC} \times \overrightarrow{AB} = (3, 6, 6)$ which will serve as our normal vector. Thus, taking as a point on the plane A, the plane is given by the equation
$$3(x-1) + 6(y-1) + 6(z-6) = 0,$$
or, simplifying,
$$x + 2y + 2z = 15.$$
(b). We define the angle between two planes as the angle between their normal vectors. Find the angle between the plan in part (a) and the plane given by the equation
$$2x + 3y + 6z = 10.$$
(You can leave your answer in terms of standard functions from your trigonometry class; i.e., a numerical approximation isn't needed).

Solution: We can take as normal vector for the above plane $\overrightarrow{n}_1 = (1, 2, 2)$ (which has norm $|\overrightarrow{n}_1| = \sqrt{1^2 + 2^2 + 2^2} = 3$), and we can read off the normal of the second one as $\overrightarrow{n}_2 = (2, 3, 6)$ (which has norm $|\overrightarrow{n}_2| = \sqrt{2^2 + 3^2 + 6^2} = 7$). Combining our two equations for the dot product gives
$$\overrightarrow{n}_1 \cdot \overrightarrow{n}_2 = (2 + 6 + 12) = 20 = |\overrightarrow{n}_1||\overrightarrow{n}_2| \cos\vartheta = 21 \cos\vartheta.$$
Thus, the angle is $\arccos(20/21)$.