UNIVERSITY OF DUBLIN

TRINITY COLLEGE

Faculty of Engineering, Mathematics and Science

school of mathematics

JF Maths/TP/TSM **Michaelmas Term 2016**

MA1111 — Linear Algebra I

Dr. Larry Rolen

Attempt all questions. All questions are weighted equally.
No calculators are permitted for this examination. All work must be shown for full credit.

1. Consider the system of linear equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ -2x_1 - 3x_2 + x_3 = -1 \\ 2x_1 + x_2 - 3x_3 = 1. \end{cases}$$

   (a) Write down an equivalent equation $Ax = b$, and find A^{-1}.

   (b) Use your answer from part (a) to solve the original system of linear equations.

Solutions: (a): Solving the system is equivalent to solving the matrix equation $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

To find A^{-1}, we can form the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{pmatrix}$$

and use elementary row operations to reduce the first three columns to the identity matrix I_3, giving

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{3} & \frac{3}{4} & \frac{11}{12} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{3}{4} & -\frac{7}{12} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{4} & \frac{1}{12} \end{pmatrix},$$

from which we read off that

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{3}{4} & \frac{11}{12} \\ -\frac{1}{3} & -\frac{3}{4} & -\frac{7}{12} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{12} \end{pmatrix}.$$

In studying for the exam, make sure that you are very quick at row reduction. Also recall that we could have also found the inverse using the adjugate matrix (using cofactors).

(b): The solution to the system is then given by $x = A^{-1}b$, which we multiply out to give

$$x = \begin{pmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ \frac{1}{6} \end{pmatrix}.$$

That is, the solution to the original system is $x_1 = 5/6$, $x_2 = -1/6$, $x_3 = 1/6$.

2. The real polynomials in one variable x of degree at most 2 are denoted by $\mathcal{P}_{\leq 2}(\mathbb{R})$.

   (a) Prove that the subset $\{1 + x, 2 + x + x^2, 4 - 3x + x^2\}$ is a basis of $\mathcal{P}_{\leq 2}(\mathbb{R})$.

   (b) Find the coordinate vector of $1 + x + 2x^2$ with respect to this basis.

Solution:

(a): A polynomial $\alpha + \beta x + \gamma x^2$ is a linear combination of $1+x, 2+x+x^2, 4-3x+x^2$ if and only if we can solve

$$a(1+x)+b(2+x+x^2)+c(4-3x+x^2) = (a+2b+4c)+(a+b-3c)x+(b+c)x^2 = \alpha+\beta x+\gamma x^2.$$

By setting each power of x on each side equal, this is equivalent to the system of equations

$$\begin{cases} a + 2b + 4c = \alpha \\ a + b - 3c = \beta \\ b + c = \gamma \end{cases}.$$

The corresponding matrix equation is

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

To show that the three polynomials in question form a basis, we need to know that they span the set of polynomials, which is the same as this matrix equation always being consistent (always having a solution), and that they are linearly independent, which is the same as saying that only the trivial linear combination (adding up 0 times each one) gives the zero polynomial. By the discussion above, this last condition is the same as saying that the kernel of

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

is the trivial subspace $\{0\}$. Thus, since this is a square matrix, we know from results in class that both of the conditions (matrix equation always has a solution and the kernel is trivial) are equivalent to $\det A \neq 0$. We thus compute, expanding along the first column:

$$\det A = \det \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} - \det \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} = 4 + 2 = 6 \neq 0.$$

Thus, the set is indeed a basis.

(b): To find the coordinate vector of a polynomial with respect to this basis, we need to express it as a linear combination of the basis polynomials. This is a special case of the discussion above: we need to solve $Ax = b$ where b corresponds to the polynomial whose coordinate vector we are looking for, namely,

$$b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

We can solve the equation by finding A^{-1} or by using row reduction. That is, we can perform row reduction on the matrix

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

to obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{3} \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{pmatrix}.$$

We thus read off that the coordinate vector in question is $(-7/3, 7/3, -1/3)$.

3. Let $f\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation defined by $f(A) = BA - A^T$, where $M_{2\times 2}(\mathbb{R})$ is the space of 2×2 real-entry matrices, and $B = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$.

   (a) Find the matrix associated to f with respect to the standard basis
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

   (b) Use your answer from (a) to find a basis for the image of f.

   (c) Use your answer from (a) to find a basis for the kernel of f.

Solution:

(a): We compute the image of each of the basis elements under the transformation:

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

The coordinate vectors of these matrices with respect to the standard basis are, respectively, $(0,0,0,0)$, $(0,1,-1,0)$, $(2,-1,1,0)$, and $(0,2,0,0)$. The matrix for f with respect to this basis has these vectors as its columns, and is thus

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b): Using row reduction, we find that the RREF of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This has the same kernel as A, and its kernel is read off by setting the only free variable, x_1 equal to an arbitrary real constant t. Then we solve for the pivotal variables x_2, x_3, x_4, giving $x_2 = x_3 = x_4 = 0$. Thus, an arbitrary element of the kernel of A is

$$\begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and so a basis for the (one-dimensional) kernel of A is given by $\{(1, 0, 0, 0)\}$. Thus, the corresponding basis for the kernel of f is the given by the set of matrices for which the elements of the basis of the kernel of A are coordinate vectors. That is, a basis for $\ker(f)$ is given by $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.

(c): Using the RREF of A we found in (b), we note that this has a pivot in columns $2, 3, 4$. The corresponding columns of A are a basis for the column space of A: $\{(0, 1, -1, 0), (2, -1, 1, 0), (0, 2, 0, 0)\}$. The corresponding matrices (for which these are coordinate vectors) is a basis for the image of f. That is, our basis is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}.$$

Although this suffices to answer the question, it is actually possible to find a much simpler basis, which you could also find by using the method we gave in class to find column space by computing the row space of the transpose (we saw that the row space of a matrix has as a basis simply the non-zeor rows in its RREF). This gives the alternative basis for the image of f:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Thus, another way of describing the image is that it is the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $d = 0$.

4. (a) Find the determinant of $A = \left(\begin{smallmatrix} -2 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{smallmatrix}\right)$ **directly from the definition we gave in class.**

   (b) Find the determinant of A^{-1}.

Solution: We defined the determinant as the unique function on rows of a square matrix which is multilinear, alternating (vanishes if two rows are equal), and takes value 1 on the identity matrix. This also implies that the determinant is zero if there is a row of all zeros, and also that swapping two rows multiplies the determinant by a sign. This is, in general all we need to compute any determinant. In this case, we find

$$\begin{aligned}
\det A &= \det((-2,0,0),(1,1,1),(3,4,1)) = \det(-2e_1, e_1+e_2+e_3, 3e_1+4e_2+e_3) \\
&= -2\det(e_1, e_1+e_2+e_3, 3e_1+4e_2+e_3) \\
&= -2\det(e_1,e_1,3e_1+4e_2+e_3) - 2\det(e_1,e_2,3e_1+4e_2+e_3) \\
&- 2\det(e_1,e_3,3e_1+4e_2+e_3) \\
&= -6\det(e_1,e_1,e_1) - 8\det(e_1,e_1,e_2) - 2\det(e_1,e_1,e_3) \\
&- 6\det(e_1,e_2,e_1) - 8\det(e_1,e_2,e_2) - 2\det(e_1,e_2,e_3) \\
&- 6\det(e_1,e_3,e_1) - 8\det(e_1,e_3,e_2) - 2\det(e_1,e_3,e_3) \\
&= -2\det(e_1,e_2,e_3) - 8\det(e_1,e_3,e_2) = -2 + 8\det(e_1,e_2,e_3) = -2+8 = 6.
\end{aligned}$$

(b): If a matrix is invertible, then the determinant of the inverse is the reciprocal of the original determinant. In this case, $\det(A^{-1}) = 1/6$.

5. (a) Given a subspace W of $\mathbb{R}^n$, prove that the *orthogonal complement* $W^\perp$, defined by $W^\perp = \{v \in \mathbb{R}^n | v \cdot w = 0 \text{ for all } w \in W\}$ is a subspace of $\mathbb{R}^n$.

   (b) Prove that if W is a subspace of $\mathbb{R}^3$, then we have the direct sum decomposition $\mathbb{R}^3 = W \oplus W^\perp$.

Solution:

(a): To check that it is a subspace, we need to verify three conditions: $0 \in W^\perp$, $v_1 + v_2 \in W^\perp$ for all $v_1, v_2 \in W^\perp$, and $cv \in W^\perp$ for all $c \in \mathbb{R}$, $v \in W^\perp$. To check the first, note that $0 \cdot w = 0$ for all $w \in W$, as in fact $0 \cdot v = 0$ for all $V \in \mathbb{R}^n$. For closure under vector addition, suppose that $v_1, v_2 \in W^\perp$, i.e., that $v_1 \cdot w = 0$, $v_2 \cdot w$ for all $w \in W$. Then for any $w \in W$, we have

$$(v_1 + v_2)w = v_1 w + v_2 w = 0 + 0 = 0,$$

so that $v_1 + v_2$ is orthogonal to all elements of W, and hence in $W^\perp$. Finally, if $v \in W^\perp$ and $c \in \mathbb{R}$, we have, for any $w \in W$,

$$(cv) \cdot w = c(v \cdot w) = c \cdot 0 = 0,$$

so that $cv \in W^\perp$. Thus, $W^\perp$ is indeed a subspace of R^n.

(b): There are several possibilities. Firstly, we could have that W is 0 or three dimensional, i.e., $W = \{0\}$ or $W = \mathbb{R}^3$. In either case, it is clear that the orthogonal complement is the other one of these two spaces (the only thing which is orthogonal to everything is the 0 vector, and everything is orthogonal to the 0 vector). It is also clear that the direct sum of $\{0\}$ and $\mathbb{R}^3$ is $\mathbb{R}^3$, as they clearly intersect only at 0 and every vector v can be written as a sum $v = 0 + v$. The other possibility is that W is a line or a plane through the origin, respectively. In this case, the orthogonal complement is the plane (resp. line) through the origin which is perpendicular to the line (resp. plane). Thus, the claim really states that given a plane through the origin and its normal line through the origin, that $\mathbb{R}^3$ is a direct sum of these two subspaces. We have seen in an explicit way how to do this for examples with numbers on the homework, so we just have to write down the same equations in general. Suppose that our plane has an equation of the shape $ax + by + cz = 0$, and then write its normal line through 0 as $\operatorname{span}\{(a, b, c)\}$. It is (geometrically, for example) clear that the plane and its normal line only intersect at the point 0. Thus, we just have to show that every point in $\mathbb{R}^3$ is a sum of a point on the plane and a point on the line. First suppose that a, b, c are all non-zero. Then a generic point on the plane is of the form $(\alpha, \beta, -\frac{a\alpha + b\beta}{c})$, and a generic point on the line is $(\gamma a, \gamma b, \gamma c)$. We want to show that any point (x, y, z) is of the form

$$(\alpha, \beta, -\frac{a\alpha + b\beta}{c}) + (\gamma a, \gamma b, \gamma c) = (\alpha + \gamma a, \beta + \gamma b, -\frac{a\alpha + b\beta}{c} + \gamma c) = (x, y, z),$$

which is equivalent to solving the matrix equation

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ -\frac{a}{c} & -\frac{b}{c} & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This is always solvable if and only if the determinant isn't zero, and by expanding along the first row, we find that its determinant is equal to

$$\det \begin{pmatrix} 1 & b \\ -\frac{b}{c} & c \end{pmatrix} + a \det \begin{pmatrix} 0 & 1 \\ -\frac{a}{c} & -\frac{b}{c} \end{pmatrix} = c + b^2/c + a^2/c = \frac{a^2+b^2+c^2}{c},$$

which can't ever be 0 when a, b, c are all non-zero. If exactly one of a, b, c is equal to 0, say without loss of generality that it is $c = 0$, and so $a, b \neq 0$. Then we just repeat the argument above. Namely, a generic point on the plane is of the form $(\alpha, -\alpha a/b, \beta,$ and a generic point on the line is $(\gamma a, \gamma b, 0)$. We want to show that any point (x, y, z) is of the form

$$(\alpha, -\alpha a/b, \beta) + (\gamma a, \gamma b, 0) = (\alpha + \gamma a, -\alpha a/b + \gamma b, \beta) = (x, y, z),$$

which is equivalent to solving the matrix equation

$$\begin{pmatrix} 1 & 0 & a \\ -\frac{a}{b} & 0 & b \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Similarly, we require that this matrix has non-zero determinant. Expanding along the bottom row, we find that it has determinant

$$-\det \begin{pmatrix} 1 & a \\ -\frac{a}{b} & b \end{pmatrix} = -(b + a/b) = -\frac{a^2+b^2}{b},$$

which again can't equal 0. If two of a, b, c are equal to 0, then again using symmetry we can assume $b = c = 0$, $a \neq 0$. Then a generic point on the plane is of the form $(0, \alpha, \beta)$ and a generic point on the normal line is $(\gamma, 0, 0)$. Thus, any point $(x, y, z) \in \mathbb{R}^3$ can be written as a sum of a point on the plane and the line as follows:

$$(x, y, z) = (0, y, z) + (x, 0, 0).$$

6. Let A be an $n \times n$ matrix satisfying $A^2 = 2A$. Find the possible values of $\det A$.

Solution:

Multiplying a row of a matrix by a constant α changes the determinant by a multiple of α. Thus, if we multiply an $n \times n$ matrix by a constant α, we are multiplying every row by α, and so the new determinant is α^n times the old one. Thus, $\det(2A) = 2^n \det A$. We also know that in general $\det(AB) = \det(A)\det(B)$, so that $\det(A^2) = \det(A)^2$. Comparing using the equation $A^2 = 2A$, we have

$$\det(A)^2 = 2^n \det(A).$$

If $\det(A) \neq 0$ we can divide both sides of the last equation by $\det(A)$, giving determinant 2^n. That is, the determinant is either 0 or 2^n.