FURTHER TYPES OF INDEFINITE THETA FUNCTIONS AND THEIR CONNECTIONS TO MAASS WAVE FORMS, QUANTUM MODULAR FORMS, AND KNOT THEORY

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1. Indefinite theta series related to Maass waveforms

In the last three lectures, we saw many examples of theta series for general indefinite quadratic forms. These were constructed to be completions of holomorphic functions (for example, built out of products of differences of sign functions) which arise in many natural areas of combinatorics, representation theory, and physics. Here, we will describe another type of indefinite theta function. Although it looks at first glance to be similar to the functions in Zwegers' thesis, we will see that it has rather different modular completions.

The first motivating example is the following q-hypergeometric series from Ramanujan's Lost Notebook:

$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q)_n} =: \sum_{n\geq 0} S(n)q^n,$$

Andrews, Dyson, and Hickerson showed that the coefficients $S(n)$ satisfy striking properties; in particular, they showed that $\limsup |S(n)| = \infty$, but at the same time $S(n) = 0$ for infinitely many n. Their proof hinged on the fact that σ can be written as an indefinite theta series:

$$\sigma(q) = \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} + \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^{n+j} q^{\frac{3}{2}(n+1/6)^2 - j^2}. \tag{1}$$

This looks very similar to the holomorphic parts of type $(1,1)$ indefinite theta series from Zwegers' thesis. Indeed, if we change a single sign and replace $\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} + \sum_{\substack{n+j<0\\ n-j<0}}\right)$ by $\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)$, we essentially get a sixth-oder mock theta function of Ramanujan.

In Section 2, we will describe an alternative theory, also due to Zwegers, for such indefinite theta series. The first instance of such a theory was given by Cohen. He used

σ, together with its complementary function

$$\sigma^\star(q) := 2\sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(q;q^2)_n},$$

to explicitly build a Maass waveform. Namely, he showed that if $(n \in 24\mathbb{Z}+1)$

$$q\sigma\left(q^{24}\right) =: \sum_{n\geq 0} T(n)q^n, \qquad q^{-1}\sigma^\star\left(q^{24}\right) =: \sum_{n<0} T(n)q^{-n},$$

then (where K_0 is a Bessel function, $\tau = x+iy$)

$$u(\tau) := y^{\frac{1}{2}}\sum_{n\in 24\mathbb{Z}+1} T(n)K_0\left(\frac{2\pi|n|y}{24}\right)e\left(\frac{nx}{24}\right)$$

transforms as a modular form on the congruence subgroup $\Gamma_0(2)$. and is an eigenfunction of the hyperbolic Laplacian Δ_0. with eigenvalue $1/4$.

Further examples of functions analogous to $\sigma(q)$ were constructed by a number of authors, such as Corson-Favero-Liesinger-Zubairy, Bringmann-Kane. and Lovejoy. Li, Ngo, and Rhoades later studied many properties of these functions, and conjectured that they all can be related to Maass waveforms like Cohen showed for $\sigma(q)$ (the main difficulty being that Cohen's method involved finding certain Hecke characters which becomes computationally infeasible for most examples). This was proven by Krauel, Woodbury, and the author by using Zwegers' theory of indefinite theta series like (1) (cf. Section 2).

It turns out that these examples also all fit into several natural, infinite families of q-hypergeometric series. This realization makes use of polynomials $H_n(k,\ell;b;q)$ (defined in (2)), which were studied in relation to torus knots and quantum modular forms (which are also described in Section 3). Specifically, the n-th coefficient in Habiro's cyclotomic expansion of the colored Jones polynomial of the left-handed torus knot $T(2,2k+1)$ was shown to be $q^{n+1-k}H_{n+1}(k,1;1;q)$, and the general $H_n(k,\ell;1;q)$ were used to construct a class of q-hypergeometric series with interesting behavior both at roots of unity and inside the unit circle. These functions, denoted by $F_j(k,\ell;q)$ $(j\in\{1,2,3,4\})$, are built from Hikami-Lovejoy's polynomials

(2)

$$H_n(k,\ell;b;q) := \sum_{n=n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\prod_{j=1}^{k-1} q^{n_j^2+(1-b)n_j}\begin{bmatrix} n_{j+1}-n_j-bj+\sum_{r=1}^{j}(2n_r+\chi_{\ell>r}) \\ n_{j+1}-n_j\end{bmatrix}_q$$

and, for example,

$$F_1(k,\ell;q) := \sum_{n\geq 0}(q)_n(-1)^n q^{\binom{n+1}{2}}H_n(k,\ell;0;q).$$

The relationship between thees series involves the following "period integral" map (cf. Exercise 3 for an explanation of this terminology), which was studied by Lewis and

Zagier and used by Zagier to show that functions like $\sigma(q)$ are quantum modular forms (cf. Section 3). Specifically, if f a Maass waveform with eigenvalue $1/4$ on a congruence subgroup which is cuspidal at $i\infty$, it has a Fourier expansion of the form

$$f(\tau) =: v^{\frac{1}{2}} \sum_{n\neq 0} A(n) K_0\left(\frac{2\pi|n|v}{N}\right) e\left(\frac{nu}{N}\right),$$

and we let

$$f^+(\tau) := \sum_{n>0} A(n) q^{\frac{n}{N}}. \tag{3}$$

Theorem (Bringmann-Lovejoy-R.)**.** *The functions $F_j\left(k,\ell;q^d\right)$ are the images of Maass cusp forms under the map* (3)*.*

Remark. *The cuspidality of the associated Maass waveforms isn't obvious. In fact, the specific shapes of the q-hypergeometric representations of F are essential.*

The idea behind the proof is that after using the Bailey pair machinery on the q-hypergeometric series defining $F_1(k,\ell;q)$, such functions become indefinite theta functions of the shape discussed in the next section. A specific example of applying Zwegers' framework to functions like F_j is sketched in Exercises 4 and 5.

2. Zwegers' Mock Maass Theta Functions

Zwegers beautifully contextualized Cohen's observations using indefinite theta series with a similar flavor as the ones in his thesis. As per our discussions in the last two lectures, finding modular completions for these series boils down to finding nice functions which satisfy Vignéras' differential equation, include the desired sums of sign functions as canonical pieces, and which yield convergent series. Of course, since the sign of series like (1) is flipped, relative to those in Zwegers' thesis, we require different functions for convergence.

As in Zwegers' thesis, the general situation begins by taking a pair of vectors $c_1, c_2 \in C_Q$. Then Q splits over $\mathbb{R}$ as a product of linear factors $Q(r) = Q_0(Pr)$ for some (non-unique) $P \in \mathrm{GL}_2(\mathbb{R})$; namely, $Q_0(r) := r_1r_2$. Then, for each $c \in C_Q$, there is a unique $t \in \mathbb{R}$ such that

$$c = c(t) := P^{-1}\begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}.$$

We also set $c^\perp = c^\perp(t) := P^{-1}\left(\begin{smallmatrix} e^t \\ e^{-t} \end{smallmatrix}\right)$. The following function then takes the role that the E function played for our original type $(1,1)$ indefinite theta series when we apply Vignéras' theorem (you will be asked to show this in Exercise 1):

$$p(v) := \int_{t_1}^{t_2} e^{-\pi B(v,c(t))^2} dt. \tag{4}$$

Using this function, Zwegers was able to complete the indefinite theta series

$$\begin{aligned}\Phi_{a,b}(\tau)=\Phi_{a,b}^{c_1,c_2}(\tau):=\operatorname{sgn}(t_2-t_1)v^{\frac{1}{2}}\sum_{r\in a+\mathbb{Z}^2}\rho_A(r)e(Q(r)u+B(r,b))K_0(2\pi Q(r)v)\\+\operatorname{sgn}(t_2-t_1)v^{\frac{1}{2}}\sum_{r\in a+\mathbb{Z}^2}\rho_A^{\perp}(r)e(Q(r)u+B(r,b))K_0(-2\pi Q(r)v),\end{aligned}$$

where

$$\rho_A(r):=\rho_A^{c_1,c_2}(r):=\frac{1}{2}\left(1-\operatorname{sgn}\left(B\left(r,c_1\right)B\left(r,c_2\right)\right)\right),$$

and $\rho_A^{\perp}:=\rho_A^{c_1^{\perp},c_2^{\perp}}$.

Assuming convergence, it is immediate from the differential equation satisfied by K_0 that $\Phi_{a,b}$ is an eigenfunction of Δ_0 with eigenvalue $1/4$. Zwegers' modular completion of $\Phi_{a,b}$ splits in terms of the functions

$$\varphi_{a,b}^{c}(\tau):=v^{\frac{1}{2}}\sum_{r\in a+\mathbb{Z}^2}\alpha_t\left(rv^{\frac{1}{2}}\right)q^{Q(r)}e(B(r,b)),$$

where

$$\alpha_t(r):=\begin{cases}\displaystyle\int_t^{\infty}e^{-\pi B(r,c(x))^2}dx & \text{if } B\left(r,c\right)B\left(r,c^{\perp}\right)>0,\\ \displaystyle-\int_{-\infty}^{t}e^{-\pi B(r,c(x))^2}dx & \text{if } B\left(r,c\right)B\left(r,c^{\perp}\right)<0,\\ 0 & \text{otherwise.}\end{cases}$$

Zwegers then showed that

$$\widehat{\Phi}_{a,b}^{c_1,c_2}:=\Phi_{a,b}^{c_1,c_2}+\varphi_{a,b}^{c_1}-\varphi_{a,b}^{c_2}$$

transforms like a modular form. Conveniently, the functions φ^c satisfy nice relationships under the action of matrices fixing the quadratic form Q, and this can be used in many examples to show that the differences of two φ functions in $\widehat{\Phi}$ cancel out. In general, this is no longer an eigenfunction for Δ_0, and hence not a Maass waveform. However, there is still a "mock" picture. Recall that completing indefinite theta functions of type $(r-1,1)$ forces one to sacrifice holomorphicity (and so are no longer in the kernel of the shadow operator ξ_k), but that the images under ξ_k of these indefinite theta functions are "nicer." In this situation, the completed function $\widehat{\Phi}$ also loses its eigenvalue property under a differential operator (Δ_0), but Zwegers showed that there is a "shadow" defined using the fact that

$$v^{-\frac{3}{2}}\left(\Delta_0-\frac{1}{4}\right)\widehat{\Phi}_{a,b}^{c_1,c_2}\in S_{\frac{3}{2}}\otimes\overline{S_{\frac{3}{2}}}.$$

3. Quantum Modular Forms

We conclude this series of lectures by giving one more way in which the above indefinite theta functions are related to modular-type objects. This is facilitated by the seemingly-innocuous map (3). It turns out that this map is a special kind of integral transform, not unlike "period integrals" which allow one to build non-holomorphic parts of harmonic Maass forms by integrating their shadows. If one carefully filters out what remnants of modularity survive this integral transformation, one discovers examples of new objects known as *quantum modular forms.* These were recently defined by Zagier and have been connected to many important combinatorial generating functions, as well as to knot and 3-manifold invariants. Roughly speaking, a *quantum modular form* is a function which is defined on $\mathbb{Q}$ and which fails to transform modular by a particularly "nice" function.

"Definition". A **quantum modular form of weight** k is a function $f\colon \mathbb{Q} \to \mathbb{C}$ such that for all γ in a congruence subgroup Γ, the "error of modularity" cocycle

$$f|_k(1-\gamma)$$

is "nice."

Clearly, this isn't a well-defined notion as stated. However, Zagier intentionally left the definition open-ended to include a wide variety of different examples, and quantum modular forms are something which you "know when you see them." To paraphrase Tolstoy, all modular-type functions (modular forms, harmonic Maass forms, etc.) are alike; each quantum modular form is quantum modular in its own way.

Zagier showed that the image of Maass waveforms under the map (3) are indeed quantum modular forms. This uses the formulation of this map as a special integral transform, and this transform and the resulting quantum modular cocycle relations are stated in Exercise 3. In the case of $\sigma, \sigma^\star$ (as well as for the families F_j above using the special properties of the polynomials of Hikami and Lovejoy), it is possible to explicitly see the values of these quantum modular forms as finite sums. To see this, we first need the identities due to Andrews and Cohen (respectively):

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (q)_n, \quad \sigma^\star(q) = -2 \sum_{n=0}^{\infty} q^{n+1} \left(q^2;q^2\right)_n . \tag{5}$$

The right hand sides of these two identities converge not only when $|q| < 1$ but also at roots of unity (as eventually $(q;q)_n$ becomes 0. Cohen further showed that (cf. Exercise 2)

$$\sigma(\zeta) = -\sigma^\star\left(\zeta^{-1}\right). \tag{6}$$

Zagier then proved the following transformation properties for function ($x \in \mathbb{Q}$)

$$f(x) := e^{\frac{\pi i x}{12}} \sigma(e^{2\pi i x}) = -e^{\frac{\pi i x}{12}} \sigma^\star(e^{-2\pi i x}).$$

Theorem (Zagier)**.** *The function $f\colon \mathbb{Q} \to \mathbb{C}$ satisfies*

$$f(x+1) = \zeta_{24} f(x), \quad f(x) - \zeta_{24}^{-1}(2x+1)^{-1} f\left(\frac{x}{2x+1}\right) = h(x),$$

where $h\colon \mathbb{R} \to \mathbb{C}$ is C^∞ on $\mathbb{R}$, and analytic on $\mathbb{R} \setminus \{-1/2\}$. In particular, f is a quantum modular form of weight 1 on the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

It turns out that related quantum modularity results also hold for period integrals arising in the theory of mock modular forms. The interested reader is referred to Zagier's original survey "Quantum Modular Forms" for a description of other such types of quantum modular forms and their relations to period integrals, Eichler integrals, and knot theory.

4. Exercises

(1) Show that the function p in (4) satisfies Vignéras' differential equation.

(2) Prove (6) using the following argument from Zagier's paper on quantum modular forms. Show that the Laurent series

$$S_k := \sum_{n=1}^{k} q^{-\frac{n(n-1)}{2}} (-q)_{k-n}$$

is a solution to the recursion

$$S_{k+1} - S_k = q^{k+1}(S_k(q) - (-q)_k) - q^{-\frac{k(k+1)}{2}}.$$

Conclude that

$$\sum_{n=0}^{k-1} (q^{-1}-1)\dots(q^{-n}-1) - \sum_{n=0}^{k-1} q^{n+1}(q^2;q^2)_n = (q)_k S_k.$$

Deduce the claimed result (6) by noting that if q is a root of unity of order at most k, then the right hand side of this equation becomes 0 and by using (5).

(3) The explicit formulation of the map (3) in terms of a "period integral", as Lewis in Zagier showed, is that

$$F^+(\tau) = -\frac{2}{\pi} \int_\tau^{i\infty} \left[F(z), R_\tau(z)\right].$$

Here, for general functions f, g which are eigenfunctions of Δ with eigenvalue $1/4$, $[f,g]$ is the Green's form

$$[f,g] := \frac{\partial f}{\partial z} g dz + \frac{\partial g}{\partial \overline{z}} f d\overline{z},$$

and ($z = x+iy$ with $x, y \in \mathbb{R}$)

$$R_\tau(z) := \frac{y^{\frac{1}{2}}}{\sqrt{(x-\tau)^2+y^2}}.$$

Show this identity. This requires some significant rewriting and a few integral transforms; a detailed proof can also be found in Proposition 3.5 of Li, Ngo, and Rhoades' paper "Renormalization and quantum modular forms, Part I: Maass wave forms." Conclude that for a Maass form F and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the congruence subgroup F is modular on, we have

$$F^+(\tau) - (c\tau + d)^{-1} F^+(\gamma\tau) = -\int_{\gamma^{-1} i\infty}^{i\infty} [F(z), R_\tau(z)].$$

After ensuring convergence towards appropriate sets of rational numbers, this formula gives the quantum modularity transformations for the quantum modular forms discussed above.

(4) Consider the q-hypergeometric series

$$f_1(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q)_n \left(1 - q^{2n+1}\right)}.$$

This series was shown by Bringmann and Kane to have the following representation:

$$f_1(q) = \sum_{-n-1\leq j\leq n} q^{4n^2+5n+1-2j^2-2j}\left(1+q^{6n+6}\right) + \sum_{-n\leq j\leq n} q^{4n^2+n-2j^2}\left(1+q^{6n+3}\right).$$

Show that f_1 fits into Zwegers' framework, where (in the notation for $\Phi_{a,b}$ above) the quadratic form has associated matrix $A = \left(\begin{smallmatrix} 8 & 0 \\ 0 & -4 \end{smallmatrix}\right)$, and where

$$a_1 = \begin{pmatrix} \frac{5}{8} \\ \frac{1}{2} \end{pmatrix}, \quad a_2 = \begin{pmatrix} \frac{1}{8} \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \text{and} \quad c_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

(5) Zwegers showed the following relations between the functions φ^{c_j} appearing in the completions of the functions Φ^{c_1,c_2}, where $a, b \in \mathbb{R}^2$:

$$\varphi^c_{a+\lambda,b+\mu} = e(B(a,\mu))\varphi^c_{a,b} \quad \text{for all } \lambda \in \mathbb{Z}^2 \text{ and } \mu \in A^{-1}\mathbb{Z}^2,$$
$$\varphi^c_{-a,-b} = \varphi^c_{a,b},$$

and

$$\varphi^{\gamma c}_{\gamma a,\gamma b} = \varphi^c_{a,b} \quad \text{for all } \gamma \in \mathrm{SO}^+(Q,\mathbb{Z}),$$

where

$$\mathrm{SO}^+(Q,\mathbb{Z}) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid Q(\gamma r) = Q(r) \text{ for all } r \in \mathbb{R}^2, \gamma(C_Q) = C_Q\right\}.$$

Use these relations to show that for the indefinite theta series f_1 from Exercise 4, the completion terms in the associated Maass form you found in that case cancel out, so that the coefficients of $f_1(q)$ encode the positive coefficients of a Maass waveform. (Hint: Use the matrix $\gamma = \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right)$.)