INDEFINITE THETA FUNCTIONS OF TYPE $(n,1)$ II: FURTHER DISCUSSION OF ZWEGERS' FUNCTIONS AND THE PROOF OF THEIR MODULARITY

LARRY ROLEN

In the last lecture, we motivated and described Zwegers' type $(n,1)$ indefinite theta functions. Several key objectives remain. Firstly, we have to describe their exact relation to the theory of harmonic Maass forms, and in particular how to compute their shadows. After all, something should really only count as a completion if it is "simpler" than the thing you started with (without some restriction like this, you could always "complete" a function by adding minus itself to it!). Finally, we have to discuss the proof. Although we will not give all details of the proof, and although we won't follow Zwegers' original proof, we shall try to emphasize the intuition behind it and explain "why" functions like $E(z)$ should arise in our theta functions. This will be crucial when we discuss more general types of indefinite theta functions in the next lecture as well.

1. Relation to harmonic Maass forms

In order to explicitly describe Zwegers' indefinite theta functions in terms of Bruinier-Funke's theory of harmonic Maass forms, we first decompose the function ρ from the last lecture. For this, note that E can be related to the beta function, which is essentially an incomplete Gamma function (as required in the Fourier expansions of harmonic Maass forms). Specifically, for all $x \in \mathbb{R}$, we have (cf. Exercise 1)

$$E(x) = \operatorname{sgn}(x)\left(1 - \beta\left(x^2\right)\right), \tag{1}$$

where for real, non-negative y

$$\beta(y) := \int_y^\infty t^{-\frac{1}{2}} e^{-\pi t} dt,$$

and we have

$$\beta(y) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \pi y\right).$$

The humble identity (1) will turn out to play several key roles for us.

For any $c_1, c_2 \in C_Q$ we may now write $\rho(n;\tau)$ as a sum of the expressions

$$\rho_j(n;\tau) = \rho_j^{c_j}(n;\tau) := (-1)^j \operatorname{sgn}\left(B(c_j,n)\right)\beta\left(-\frac{B(c_j,n)^2}{Q(c_j)}v\right),$$

$$\rho_3(n;\tau) = \rho_3^{c_1,c_2}(n;\tau) := \operatorname{sgn}\left(B(c_1,n)\right) - \operatorname{sgn}\left(B(c_2,n)\right).$$

Specifically, we may decompose ρ as

$$\rho(n;\tau)=\begin{cases}\rho_1(n;\tau)+\rho_2(n;\tau)+\rho_3(n;\tau) & \text{if } c_1,c_2\in C_Q,\\ \rho_1(n;\tau)+\rho_3(n;\tau) & \text{if } c_1\in C_Q\ c_2\in S_Q,\\ \rho_2(n;\tau)+\rho_3(n;\tau) & \text{if } c_1\in S_Q\ c_2\in C_Q,\\ \rho_3(n;\tau) & \text{if } c_1,c_2\in S_Q.\end{cases}$$

Then we define for any $c_1, c_2 \in \overline{C}_Q$ the series

$$\Theta^+_{a,b,c_1,c_2}(\tau)=\Theta^+_{A,a,b,c_1,c_2}(\tau):=\sum_{n\in a+\mathbb{Z}^r}\rho_3(n;\tau)e(B(n,b))q^{Q(n)},$$
$$\Theta^-_{a,b,c_1,c_2}(\tau)=\Theta^-_{A,a,b,c_1,c_2}(\tau):=\sum_{n\in a+\mathbb{Z}^r}\left(\rho(n;\tau)-\rho_3(n;\tau)\right)e(B(n,b))q^{Q(n)}.$$

The point of this definition, as the notation suggests, is that Θ^+_{a,b,c_1,c_2} is the holomorphic part of Θ_{a,b,c_1,c_2}, and, similarly, Θ^-_{a,b,c_1,c_2} is the non-holomorphic part.

We want to discuss exactly how we can see Θ^+ as a mixed mock modular form and how to compute its shadow. For this, we require special non-holomophic functions, which are like those used in the construction of Zwegers' $\widehat{\mu}$. For any $a, b \in \mathbb{R}$ and $\tau \in \mathbb{H}$, let

$$R_{a,b}(\tau):=\sum_{n\in a+\mathbb{Z}}\operatorname{sgn}(n)\beta\left(2n^2v\right)e(-nb)q^{-\frac{n^2}{2}}.$$

Recall that $g_{a,b}$ denotes a special unary theta function

$$g_{a,b}(\tau):=\sum_{n\in a+\mathbb{Z}}ne(nb)q^{\frac{n^2}{2}}.$$

Similarly to the case of the R function from Chapter 1 of Zwegers' thesis, one can simply differentiate to show the following, where

$$\xi_k:=2iv^k\overline{\frac{\partial}{\partial\overline{\tau}}}$$

is the *shadow operator* of weight k (Exercise 2 will ask you to show this):

$$\xi_{\frac{r}{2}}\left(R_{a,b}\right)=-v^{\frac{r-1}{2}}\sqrt{2}g_{a,b}. \tag{2}$$

The non-holomoprhic parts (or equivalently, shadows) have shapes dictated by the following decompositions. If $c_j \in \overline{C}_Q \cap \mathbb{Z}^r$ has relatively prime coordinates, then we write $n = \ell + Nc_j$ with $\ell \in a + \mathbb{Z}^r$, $N \in \mathbb{Z}$, such that $\frac{B(c_j,\ell)}{2Q(c_j)} \in [0, 1)$. As $c_j \in \mathbb{Z}^r$, we may write the set of such ℓ as a disjoint union

$$\left\{\ell\in a+\mathbb{Z}^r:\frac{B(c_j,\ell)}{2Q(c_j)}\in[0,1)\right\}=\dot{\bigcup}_{\ell_0\in P_{0,j}}\left(\ell_0+\langle c_j\rangle^{\perp}_{\mathbb{Z}}\right),$$

where $P_{0,j}$ is a suitable finite set and where $\langle c_j\rangle_{\mathbb{Z}}^{\perp} := \{\lambda \in \mathbb{Z}^r : B(c_j,\lambda)=0\}$. Then the following can be shown by a direct computation, where $\ell_0^{\perp} := \ell_0 - \frac{B(c_j,\ell_0)}{2Q(c_j)}c_j$ and

$$\mathcal{J} := \{j \in \{1,2\} : c_j \in C_Q\}.$$

Proposition 1. *Suppose that $c_1, c_2 \in \overline{C}_Q$ and that $c_j \in \mathbb{Z}^r$ have relatively prime coefficients. Then we have*

$$\xi_{\frac{r}{2}}\left(\Theta_{a,b,c_1,c_2}(\tau)\right) = \sqrt{2}v^{r-1} \times \sum_{j\in\mathcal{I}} (-1)^j \sum_{\ell_0\in P_{0,j}} g^c_{\frac{B(c_j,\ \ell_0)}{2Q(c_j)}, B(c_j,b)}\left(-2Q\left(c_j\right)\tau\right) \overline{\sum_{\nu\in\ell_0^{\perp}+\langle c_j\rangle_{\mathbb{Z}}^{\perp}} e\left(B\left(\nu,b^{\perp}\right)\right) q^{Q(\nu)}}.$$

Here, the ^{c} denotes taking complex conjugation of the Fourier coefficients. The rightmost sum in the last equation is a theta function on a shifted $(r-1)$-dimensional lattice, and so in particular is a modular form of weight $(r-1)/2$. Thus, the shadow has a simple representation in terms of ordinary modular forms.

Example 1. Andrews provided a "Hecke-type double sum" representation of Ramanujan's fifth order mock theta function f_0, which, after rewriting, yields

$$f_0(q) = \frac{1}{(q)_\infty}\left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{\frac{5n^2}{2}+\frac{n}{2}-j^2}.$$

A short calculation then shows that we may write this in terms of the type $(1,1)$ indefinite theta function

$$\frac{q^{\frac{1}{60}}}{2\eta(\tau)}\sum_{n\in a+\mathbb{Z}^2}\left(\operatorname{sgn}(B(n,c_1)) - \operatorname{sgn}(B(n,c_2))\right) q^{Q(n)} e\left(B(n,b)\right),$$

where

$$Q(j,k) := \frac{1}{2}\left(5j^2 - 2k^2\right),\ a := \begin{pmatrix}\frac{1}{10}\\ 0\end{pmatrix},\ b := \begin{pmatrix}0\\ \frac{1}{4}\end{pmatrix}, c_1 := \begin{pmatrix}2\\ 5\end{pmatrix},\ c_2 := \begin{pmatrix}-2\\ 5\end{pmatrix}.$$

As $c_1, c_2 \in C_Q$, we can apply Zwegers' theory, and we find that

$$f_0(q) = \frac{q^{\frac{1}{60}}}{2\eta(\tau)}\Theta^+_{a,b,c_1,c_2}(\tau).$$

Exercise (3) will ask you to compute the shadow of this mock theta function. Similar examples can be used to show mock modularity and compute the shadows of all of Ramanujan's original mock theta functions.

Example 2. Sometimes, indefinite theta functions have enough internal symmetries to become ordinary modular forms. For example, indefinite theta functions on higher dimensional lattices can be built out of products of simpler (in)-definite theta series. We will see an important example of such a decomposition in the next lecture, with an application to Gromov-Witten theory. For now, we record the following well-known Hecke-type expansion for the weight 1 cusp form

$$\eta(\tau)^2 = \left(\sum_{n,m\geq 0} - \sum_{n,m<0} \right) (-1)^{n+m} q^{\frac{n^2}{2}+2nm+\frac{m^2}{2}+\frac{1}{2}n+\frac{1}{2}m+\frac{1}{12}}.$$

This can be seen by writing down the associated Artin representation, along the lines of (and in fact related to) the discussion of Cohen's results the indefinite theta series $\sigma, \sigma^\star$ studied in the next lecture.

2. Vignéras' theorem and the proof of modularity of Zwegers' functions

The main theoretical tool for showing modularity of theta series such as Zwegers' is the following result of Vignéras, which determines the appropriate conditions for multiplicative factors to add into the theta functions while retaining modularity.

Lemma. *Let Q be an integral quadratic form of type $(r-s, s)$, and suppose $p\colon \mathbb{R}^r \to \mathbb{C}$ satisfies the following differential equation for some $\lambda \in \mathbb{Z}$*

$$\left(\mathcal{E} - \frac{\Delta_Q}{2\pi}\right) p = \lambda p, \tag{3}$$

where $\mathcal{E}$ is the Euler operator

$$\mathcal{E} := \sum_{j=1}^{r} x_j \frac{\partial}{\partial x_j}.$$

Assuming convergence, the theta function

$$\Theta_{A,a}(z;\tau) = v^{-\frac{\lambda}{2}} \sum_{n\in a+\mathbb{Z}^n} p\left(\sqrt{v}\left(n+\frac{y}{v}\right)\right) e(B(n,z)) q^{Q(n)}$$

is a vector-valued Jacobi form of weight $r/2+\lambda$.

How might one find nice solutions to this differential equation? It is difficult to find all solutions of large-order differential equations, but there is a trick to relate finding solutions of Vigéras' equation to a one-dimensional problem. Specifically, one can show that if c is a fixed vector with $Q(c)<0$, then a function

$$g = f\left(\frac{B(x,c)}{\sqrt{-Q(c)}}\right)$$

satisfies the differential equation for a one-variable function $f(z)$ if

$$zf' - f''/(2\pi) \tag{4}$$

returns a multiple of f (cf. Exercise 4). This is almost the famous differential equation

$$y'' + 2xy' = 2ny,$$

whose solutions are known. In particular, they can be written in terms of iterated integrals of the complementary error function erfc, which when translated correctly returns our error function $E(z)$ as a solution. Another family of solutions is given by the *Hermite polynomials*, which are discussed in relation to indefinite theta series in Stopples' paper "Theta series for indefinite quadratic forms over real number fields." These Hermite polynomials are neatly defined by can be defined in terms of iterated derivatives by

$$\mathcal{H}_m(z) := e^{2\pi z^2} \frac{\partial^m}{\partial z^m} \left[e^{-2\pi z^2} \right].$$

We thus have an idea where the function E comes from, and what makes it special. By (1), it also has the right shape to produce harmonic functions (with respect to the weight k Laplacian) when plugged into Vignéras' framework. Doing so directly will fail, as one gets a divergent series (indeed, the sign term in (1) doesn't decay at alltowards infinity). However, Zwegers made the brilliant observation that the *difference* of two such functions does. Although this proof takes significant effort (and is one of the more difficult parts of his proof as a whole), we can get an inkling of what is going on using our decompositions above. The function β in (1) decays as $y \to \infty$, so in some sense the non-holomorphic part should have an easier time converging. The holomorphic part, on the other hand, inherits a difference of sign functions (note that by definition the sign function occurs in every case regardless of whether $c_j \in S_Q$ or not). Thus, in the summation over the lattice, only certain terms do not vanish in this difference of sign functions, and the theta function is only really summing over a cone. If this cone is a region where Q is positive-definite, our theta series will then be convergent, and it will be modular as we have still summed the original E function over the whole lattice. A very simple example of such convergence, which you can play with by hand, is given in Exercise 7.

Finally, we have to make a remark about the cuspidal case. If $c_j \in S_Q$, then the discontinuities of the sign function present a problem. Here, we must actually express this as a limit of vectors $c_{j,t}$ and prove modularity of the limit of Vignéras theta functions. This is facilitated, once again, by the fact that the sign function is the limiting function remaining as $y \to \infty$ in (1).

3. Exercises

(1) Show (1) by writing $\int_0^z e^{-\pi t^2} dt$ as $\operatorname{sign}(z) \int_0^{|z|} e^{-\pi t^2} dt$ and subsituting $t = \sqrt{u}$.
(2) Prove (2).
(3) Compute the shadow of the fifth order mock theta function f_0 in Example 1 (if you need a hint, an equivalent expression is given on page 77 of Zwegers' thesis).
(4) Fill in the details for the claimed relation of the differential equation (4) to Vignéras' differential equation.

(5) Let
$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
and $c \in \mathbb{R}^3$ such that $Q(c) = -1$. Write $x \in \mathbb{R}^3$ as $x = (x_1, x_2, x_3)$. The associated bilinear form is $(n = (n_1, n_2, n_3))$
$$B(r, n) = n_1 x_1 - n_2 x_2 - n_3 x_3.$$
In this case, the associate Laplacian is
$$\Delta_Q = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},$$
and the Euler operator is
$$\mathcal{E} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$
Then show that
$$[\mathcal{E}, \Delta] = -2\Delta.$$
Show that if p satisfies Vignéras' equation with eigenvalue d, we thus have that $\Delta_Q p$ satisfies it with eigenvalue $d - 2$.

(6) A function $f \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called *positive homogenous of degree k* if
$$f(tx) = t^k f(x) \tag{5}$$
for all $t > 0$. Show *Euler's homogenous function theorem*, which states that a $\mathcal{C}^1$ function f is positive homogenous of degree k if and only if
$$\mathcal{E}(f) = kf.$$
Hint: Differentiate both sides of (5) with respect to t and substitute $t = 1$. Conversely, integrate.

In particular, this shows that spherical polynomials (recall that these satisfy $\Delta_Q p = 0$ and are assumed to be homogenous) satisfy Vignéras' equation, confirming our earlier claim that (positive definite) spherical theta functions are modular (as long as they converge, which is clear due to the exponential decay of terms $q^{Q(n)}$).

(7) Consider the very simple type $(1, 1)$ form $Q(x_1, x_2) := x_1^2 - x_2^2$, and let $c_1 = (1, 2)$, $c_2 = (1, 3)$, $a = (1/3, 0)$, $b = (0, 0)$. Then with $x = (n + 1/3, j) \in 1/3 + \mathbb{Z}^2$, we have (you should write this down explicitly to check)
$$\Theta^+_{A,a,b,c_1,c_2}(\tau) = \sum_{n,j \in \mathbb{Z}} \left(\operatorname{sign}(-4j + 2n + 2/3) - \operatorname{sign}(-6j + 2n + 2/3)\right) q^{(n+1/3)^2 - j^2}.$$
The only terms that are non-vanishing in this sum are those for which
$$\operatorname{sign}(-4j + 2n + 2/3) = -\operatorname{sign}(-6j + 2n + 2/3)$$

(note that neither is ever 0 due to the presence of the $2/3$). Show that if this last condition holds, then $Q(n + 1/3, j) > 0$, and conclude that $\Theta^+_{A,a,b,c_1,c_2}(\tau)$ is convergent.