Math 2106-D, Foundations of Mathematical Proof
SAMPLE FINAL EXAM

Solve 6 of the following 7 problems. Circle the numbers of the problems which are to be graded.

1. Prove or disprove each of the following.
   (a) For all sets A, B, and C, $(A \cup B) - C = A \cup (B - C)$.
   (b) For all sets A, B, and C, $A - (B \cap C) = (A - B) \cup (A - C)$.
2. Prove or disprove each of the following.
   (a) If a and b are real numbers such that the product ab is an irrational number, then either a or b must be an irrational number.
   (b) If $2^n - 1$ is prime, then n must also be prime (hint: consider the identity $a^x - 1 = (a-1)(a^{x-1} + a^{x-2} + \ldots + a + 1)$).
3. (a) Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.
   (b) Prove that for every positive real number x,
   $$\frac{x}{x+1} < \frac{x+1}{x+2}.$$
4. Define a relation R on $\mathbb{R}$ by saying that xRy if and only if $x^m = y^n$ for some $m, n \in \mathbb{N}$. Show that this is an equivalence relation. Describe the possible sizes of equivalence classes of this equivalence relation.
5. (a) Show that given any real numbers a, b, c, d with $a < b$ and $c < d$, the intervals (a, b) and (c, d) have the same cardinality.
   (b) Use the Cantor-Schröder-Bernstein Theorem to show that $[0, 1]$ has the same cardinality as $(0, 1)$.
6. Suppose that G is a group, and that H and K are subgroups of G.
   (a) Show that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.
   (b) Show that $H \cap K$ is always a subgroup of G.
7. Let $x_1 = 1$ and
   $$x_{n+1} = \frac{n}{n+1} x_n^2.$$
   (a) Show that $\lim_{n\to\infty} x_n$ exists.
   (b) Show that $\lim_{n\to\infty} x_n = 0$.