Math 2106-D, Foundations of Mathematical Proof
SAMPLE EXAM 2

1. Show that if r is a real number not equal to 1, then for every $n \geq 0$, we have

$$r^0 + r^1 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

2. For any real number a, let I_a be the interval $[a, a+1)$. Describe an equivalence relation R on $\mathbb{R}$ whose equivalence classes are exactly the sets I_a for all $a \in \mathbb{Z}$ (be sure to prove that the relation you write down actually is an equivalence relation).

3. For any $n \in \mathbb{N}$, a *partition* of n is a representation of n as a sum of a non-inreasing sequence of natural numbers. For example, the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n. For example, as there are 5 partitions of 4, we have $p(4) = 5$. The number of partitions of n grows very rapidly. Show that this is true by showing that for all $n \geq 2$, we have

$$p(n) \geq 2^{\lfloor \sqrt{n} \rfloor},$$

where $\lfloor x \rfloor$ is the *floor function* which takes any real number x to the greatest integer less than or equal to x. For instance, $\lfloor \pi \rfloor = \lfloor 3.9999 \rfloor = \lfloor 3 \rfloor = 3$. (Hint: Construct a surjection from the set of partitions of n to the power set of $\{1, 2, \ldots \lfloor \sqrt{n} \rfloor\}$.)

4. Define a relation R on $\mathbb{N}$ by declaring that xRy if and only if $x = y\alpha^2$ for some $\alpha \in \mathbb{Q}$. Show that R is an equivalence relation. Show that each equivalence class has the same cardinality as $\mathbb{N}$.

5. Given a function $f\colon A \to B$ and subsets $Y, Z \subseteq B$, show that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$. (Recall that $f^{-1}(Y)$ for a general function means the preimage of Y, even when there is no inverse function f^{-1}. In the case when f is a bijection, then the preimage of Y is equal to the image of Y under f^{-1}, also denoted by $f^{-1}(Y)$).