Math 2106-D, Foundations of Mathematical Proof
Quiz (ungraded) **Solution**
November 21, 2017

Compute the following limit in two ways: using the limit theorems and directly from the definition of limits.

$$\lim_{n\to\infty} \frac{n-1}{n+1}.$$

Solution:

By using the limit rules, we find that

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)} = \frac{\lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}} = \frac{1-0}{1+0} = 1.$$

To prove this directly from the definition, let $\varepsilon > 0$. We look at the distance between the n-th term in the sequence, $(n-1)/(n+1)$, and the claimed limit, 1,

$$\left|\frac{n-1}{n+1} - 1\right|.$$

We simplify this expression using algebra;

$$\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{n-1-n-1}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1}.$$

Now

$$\frac{2}{n+1} < \varepsilon \iff n > \frac{2}{\varepsilon} - 1.$$

Thus, for any $\varepsilon > 0$, if $n > N = 2/\varepsilon - 1$, then $\left|\frac{n-1}{n+1} - 1\right| < \varepsilon$. By the definition of the limit,

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$