Math 2106-D, Foundations of Mathematical Proof
Homework 3
Due September 14, 2017

Do the following problems from Hammack:

Chapter 5: 6,12,18,28,30

Chapter 6: 10,14,22

Also turn in the following exercises:

A1 Show that there are infinitely many primes of the form $4n + 3$, that is, which are congruent to 3 modulo 4. (Hint: Use a proof by contradiction and modify Euclid's proof that there are infinitely many prime numbers from class. This time, instead of considering the product of all the prime numbers and adding one, suppose that there are finitely many prime numbers of the form $4n+3$, say $p_1, \ldots, p_k$. Then consider the number $4p_1 \cdot p_2 \cdot \ldots \cdot p_k - 1$.)

A2 You have seen that the *harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges. The *n-th harmonic number* is the finite sum

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} = \sum_{j=1}^{n} \frac{1}{j}.$$

Show, by way of contradiction, that H_n is not an integer for any natural number $n \geq 2$.

(Hint: Consider the 9-th harmonic number H_9. If it were an integer, call it n, then we would have

$$n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}.$$

The largest power of 2 dividing any of the denominators is $8 = 2^3$. Multiplying by the next lowest power of 2, namely $4 = 2^2$, and rearranging, gives

$$-\frac{1}{2} = 1 + 2 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{4}{9} - 4n.$$

Figure out what this says that $-\frac{1}{2}$ is equal to a rational number of the form $\frac{a}{b}$ for integers a, b with b odd. Why is this a contradiction? Write down a generalization and more detailed version of this argument for any natural number n.)