Math 2106-D, Foundations of Mathematical Proof
EXAM 2 SOLUTIONS
October 31, 2017

1. Show that for all integers $n \geq 2$, the following product formula holds:
$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Solution: We proceed by induction. In the base case, when $n = 2$, we check that $1-\frac{1}{4}=\frac{3}{4}$. Now suppose that the formula holds for a fixed but arbitrary natural number $n \geq 2$. Then the left hand side of the formula at $n+1$ is, by the inductive hypothesis,
$$\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{(n+1)^2}\right)=\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)\cdot\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right).$$
Upon simplifying, this becomes
$$\frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\cdot\left(\frac{(n+1)^2-1}{(n+1)^2}\right)=\frac{n^2+2n}{(n+1)\cdot(2n)}=\frac{n(n+2)}{(n+1)\cdot(2n)}.$$
This last expression is just $\frac{n+2}{2n+2}=\frac{(n+1)+1}{2(n+1)}$, which is the right hand side of the claimed formula at $n+1$. Thus, by induction, the formula holds for all $n \geq 2$.

2. Let $X = \{(x, y) \in \mathbb{Z}\times\mathbb{Z} \mid y \neq 0\}$. Define a relation R on X by declaring that $(a, b)R(c, d)$ if and only if $ad = bc$. Show that this is an equivalence relation.

Solution: This is reflexive as $ab = ba$ for any integers a, b, and so $(a, b)R(a, b)$. This is symmetric as if $(a, b)R(c, d)$, then $ad = bc$ which implies that $cb = bc = ad = da$. But this means by definition that $(c, d)R(a, b)$. Finally, this is transitive as if $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $ad = bc$, so that $adf = bcf$, and $(c, d) = (e, f)$, so that $cf = de$. Plugging this into the last equation, we find that $adf = b(de)$. Cancelling out the d's (recall that by definition the second component of each pair is non-zero), we find that $af = be$. But this is equivalent to $(a, b)R(e, f)$. Thus, this is an equivalence relation.

3. Suppose that the following fraction is a real number:
$$\frac{N\times E\times W\times Y\times O\times R\times K}{C\times H\times E\times E\times S\times E\times C\times A\times K\times E},$$
where each letter denotes an integer in $\{0, 1, \ldots, 9, 10\}$. If every letter represents a different congruence class modulo 11, find the value of this fraction (Hint: Consider the corresponding function from letters to numbers. What is the range of this function? Use the Pigeonhole Principle to prove what the range of this function is.).

Solution: This assignment describes a function from the set of letters appearing in the fraction (of which there are 11) to the set of integers in $\{0, \ldots, 10\}$, which also has 11 elements. The statement of the problem specifies that this function is an injection. We claim that its also a surjection. Indeed, if it were not, then since the codomain has the same size as the domain, the range of this function would be strictly smaller than the size of the domain. That is, this would give an injective function with smaller codomain than (finite) domain. The Pigeonhole Principle is exactly the statement that this is impossible. As the function is surjective, one letter represents 0. We know this fraction is a real number, so the 0 must be in the numerator, and so the fraction is 0.

4. Show that if A and B are any two sets with $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$ (hint: given a bijection from A to B, consider the function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ which takes a subset of A to its image under f).

   Solution: We recall that $|A| = |B|$ means that there is a bijection from A to B; thus, let $f\colon A \to B$ be a bijection. We will construct a bijection F from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Given an element $X \in \mathcal{P}(A)$, that is, a subset $X \subseteq A$, define

   $$F(X) = f(X) \in \mathcal{P}(B).$$

   We aim to show that F is a bijection. To show that it is injective, suppose that $F(X) = F(Y)$. Then we want to show that $X = Y$. To show this, it suffices to show that $X \subseteq Y$ and $Y \subseteq X$. We will just show the former, as this is symmetric in X and Y. Let $x \in X$. Then $f(x) \in F(X) = F(Y)$, and so $f(x) = f(y)$ for some $y \in Y$. But f is an injection, so $f(x) = f(y)$ implies that $x = y$. Then we also have that $x \in Y$. As X was arbitrary, we've shown that $X \subseteq Y$. Alternatively, we could directly deduce that $X = Y$ by applying the inverse function f^{-1} to both sides of the equality $f(X) = f(Y)$ (which exists since f is bijective, but note that we have proved a slightly stronger result than is required here as we only used the injectivity of f).

   To show that F is surjective, suppose that $Y \subseteq B$. Then a preimage for Y under F is $f^{-1}(Y)$. The is, since f is a bijection, the preimage is actually just the image of Y under the inverse function f^{-1}. But then the image under f of the image under f^{-1} is the same as Y itself, as any element of y gets mapped under this composition to $f(f^{-1}(y)) = \mathrm{id}_B(y) = y$.

   Alternatively, we could show that there is also an injection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and invoke the Cantor-Bernstein-Schröder Theorem. This injection in the opposite direction follows in the same way, since f is a bijection and so we can consider the bijection f^{-1} going the opposite direction and run the same argument above.

5. Show that there is a bijection from $\mathbb{Q}^5$ to the set of real numbers which are roots of polynomials of the form $ax^2 + bx + c$ for $a, b, c \in \mathbb{Z}$ (you may assume facts proven in lecture, as well as the fact that the union of a countable collection of countable sets is countable, but you may not assume any facts which you proved on homework problems).

   Solution: We claim that both of these sets are countably infinite, which would show that both sets have the same cardinality as $\mathbb{N}$, and hence as each other. As both sets are infinite, it suffices to show that both are countable. We know that $\mathbb{Q}$ is countable, and that Cartesian products of countable sets are countable, and so $\mathbb{Q}^5$ is countable too. The set of real roots of a polynomial degree at most 2 has cardinality at most 2, and is in particular finite. Now there are countably many polynomials $ax^2 + bx + c$ with integral coefficients, as the function from the set of these polynomials to $\mathbb{Z}^3$ which sends $ax^2 + bx + c$ to (a, b, c) is clearly a bijection, and by the same reasoning as above for $\mathbb{Q}^3$, this is thus countable (as $\mathbb{Z}$ is countable, being in particular a subset of $\mathbb{Q}$). The set of all real roots of all such polynomials is thus a countable union of countable (even finite) sets, and hence countable as well.