Math 2106-D, Foundations of Mathematical Proof
EXAM 1
September 21, 2017

SOLUTIONS

All problems are worth 10 points. To obtain full credit, you **must** show your work and write clearly in complete sentences. No calculators or other electronic devices are allowed.

1. Suppose that P, Q, and R are logical statements. Fill out a complete truth table to show that $\sim(P \vee Q \vee R)$ is logically equivalent to $(\sim P) \wedge (\sim Q) \wedge (\sim R)$.

   Solution: We first fill in the following truth table:

P	Q	R	$\sim P$	$\sim Q$	$\sim R$	$(\sim P) \wedge (\sim Q) \wedge (\sim R)$	$P \vee Q \vee R$	$\sim(P \vee Q \vee R)$
T	T	T	F	F	F	F	T	F
T	T	F	F	F	T	F	T	F
T	F	T	F	T	F	F	T	F
T	F	F	F	T	T	F	T	F
F	T	T	T	F	F	F	T	F
F	T	F	T	F	T	F	T	F
F	F	T	T	T	F	F	T	F
F	F	F	T	T	T	T	F	T

   As the 7-th and 9-th columns have the same entries for every possible truth value of the variables P, Q, R, the corresponding statements are logically equivalent, as claimed.

2. Suppose that $x \in \mathbb{R}$. Show that if x^2 is irrational, then x is also irrational.

   Solution:

   We will show this using the contrapositive. Thus, suppose x rational. Then we can write $x = a/b$ for integers a and b with $b \neq 0$. Squaring this gives $x^2 = a^2/b^2$, which is a ratio of two integers (with non-zero denominator), and thus rational. This establishes the contrapositive, and hence the original claim is true.

3. Show that if A and B are sets whose power sets are equal, that is, for which $\mathcal{P}(A) = \mathcal{P}(B)$, then we have $A = B$. (Hint: Use the fact that any set is a subset of itself.)

   Solution:

   Suppose that $\mathcal{P}(A) = \mathcal{P}(B)$. Then since any set is a subset of itself, we have $A \subseteq A$. In terms of the power set, this means that $A \in \mathcal{P}(A) = \mathcal{P}(B)$. Again using the definition of power sets, $A \subset B$. By the same reasoning, $B \subseteq A$. Hence, $A = B$, as desired.

4. Show that there are infinitely many prime numbers of the form $6n + 5$ for $n \in \mathbb{N} \cup \{0\}$ (Hint: Modify Euclid's proof that there are infinitely many primes. Instead of considering the number $N = p_1 \cdot \ldots \cdot p_k + 1$ as in Euclid's proof, consider the number $N = 6 \cdot p_1 \cdot \ldots \cdot p_k - 1$).

   Suppose that there are only finitely many primes of the form $6n + 5$. Then we can list them as $p_1, \ldots, p_k$, where k is the number of such primes. Consider the number

$N = 6 \cdot p_1 \cdot \ldots \cdot p_k - 1$. Then no prime $p \in \{p_1, \ldots, p_k\}$ divides N, as if it did, then $p|(6 \cdot p_1 \cdot \ldots \cdot p_k)$ and $p|N$. But then p also divides the difference of these two numbers, which is 1, and this isn't possible for an integer which is at least 2.

Thus, N is not divisible by any prime on the list of *all* primes which are 5 modulo 6. Combining with the Fundamental Theorem of Arithmetic, we find that N factors as a product of primes $q_1, \ldots, q_\ell$ which are all congruent to $0, 1, 2, 3$, or 4 modulo 6. If an integer is congruent to $0, 2$, or 4 modulo 6, then it is even, and the only even prime is 2. Similarly, the only prime which is 3 mod 6 is 3 itself (as any number which is 3 mod 6 is divisible by 3). Thus, $N = q_1 \cdot \ldots \cdot q_\ell \equiv 1 \cdot \ldots \cdot 1 \cdot 2^m 3^n \pmod{6}$, where $m, n \geq 0$ are the powers of 2 and 3 appearing in the prime factorization of N. Thus, N is congruent to 0 (if both $m, n > 0$), 1 (if $m = n = 0$)), 2 (if $n = 0$ and m is odd), 3 (if $m = 0$ and $n > 0$), or 4 (if $n = 0$ and m is positive and even), modulo 6. In particular, N can't be congruent to 5 mod 6. But from the definition of N, we find that $N \equiv -1 \equiv 5 \pmod{6}$. This is a contradiction. Thus, there are infinitely many primes of the form $5n + 6$.

5. Show if x is a positive real number, then

$$x + \frac{4}{x} \geq 4.$$

Solution:

As x is positive, we can multiply both sides of the inequality to find that $x + \frac{4}{x} \geq 4$ if and only if $x^2 + 4 \geq 4x$, which is equivalent to $x^2 - 4x + 4 \geq 0$. Noting that $x^2 - 4x + 4 = (x - 2)^2$, and that the square of any real number is positive, we find that the last inequality holds for all x. As this inequality is true for positive x if and only if $x + \frac{4}{x} \geq 4$ (both of the reductions to this inequality were if and only if statements), this holds for all positive x as well.