Further applications of Jordan Can. Form!

As we saw, JCF is the "closest" a matrix can be transformed via similar matrices to being diagonal.

Diagonalizable is very handy for many things!

Determinants are easy, so are matrix powers.

Ex: Fibonacci #s: 0, 1, 1, 2, 3, 5, 8, ...

$F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.

record recursion as: $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$, set $X := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = X^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Rightarrow$ $F_n = (0\ 1)\, X^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

To compute X^n, we diagonalize. We saw this in JCF:

$\mathrm{Char}(X) = \begin{vmatrix} x-1 & 1 \\ 1 & x \end{vmatrix} = x^2 - x - 1 \rightsquigarrow$ roots (eigenvalues)

$\lambda = \dfrac{1 \pm \sqrt{5}}{2}$ Golden ratio + conjugate real

Set $\varphi := \dfrac{1+\sqrt{5}}{2}$, $\psi := \dfrac{1-\sqrt{5}}{2}$. $\to \mathrm{Char}(X) = (x-\varphi)(x-\psi)$ elem. divisors

Then X is similar to $D = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}$ (simple eigenvalues)

We discussed how the book gives an algorithm to find P s.t. $X = PDP^{-1}$, but we can also do so directly.

To so, we compute eigenvectors for $\lambda = \varphi, \psi$.

~~Setting~~ If we find v_1, v_2 s.t.: $Xv_1 = \varphi v_1$, $Xv_2 = \psi v_2$,

setting $P := \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$, $XP = (Xv_1 \; Xv_2) = \begin{pmatrix} | & | \\ \varphi v_1 & \psi v_2 \\ | & | \end{pmatrix}$

$= (v_1 \; v_2)\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} = \cancel{PD \Rightarrow X = D}$

$= PD \Rightarrow X = PDP^{-1}$.

This last step is allowed since the eigenvalues are distinct
$\Rightarrow$ the eigen ~~values~~ vectors are lin. independent (clear in 2D case).
$\Rightarrow P$ is invertible.

Returning to Fibonacci, to compute eigenvectors, we compute null spaces of $\varphi I - X$, $\psi I - X$ via row operations (e.g.: $\varphi I - X = \begin{pmatrix} \varphi-1 & -1 \\ -1 & \varphi \end{pmatrix} = \begin{pmatrix} -\psi & -1 \\ -1 & \varphi \end{pmatrix}$

$\xrightarrow{R_1 - \psi R_2} \begin{pmatrix} 0 & 0 \\ -1 & \varphi \end{pmatrix} \xrightarrow[-R_1]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -\varphi \\ 0 & 0 \end{pmatrix} \Rightarrow$ null space spanned by $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$, take $v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$).

Then one computes $F_n = (0\;1)X^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0\;1)(PDP^{-1})^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= (0\;1)PD^nP^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0\;1)P\begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix}P^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underset{\text{(small calc.)}}{=} \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$ (Binet's Formula).

$= \dfrac{\varphi^n - \psi^n}{\varphi - \psi}$

Cor: (First noted by Kepler)

$\lim_{n\to\infty} F_{n+1}/F_n = \varphi$.

Other app's of taking powers:

Markov chain: System where member of set change membership to several states over time, probability in each step is <u>independent</u> of time (special stochastic process)

<u>Ex</u>: If its sunny, say in a model 80% chance sunny next day, 20% cloudy next day. If its cloudy, 50% chance sunny/cloudy next day

<u>Transition matrix</u>: $P = \begin{pmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{pmatrix}$ Columns consist of pos. #'s adding to 1.

sunny day rep. by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. After one day,

$P \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$ representing <u>probabilities</u>.

What happens after many days / on average?

char poly $(P) = x^2 - \operatorname{tr}(P)x + (-1)^2 \det(P) = x^2 - 1.3x + 0.3$

$\rightsquigarrow$ eigenvalues $\lambda = 1, 0.3 \rightsquigarrow P$ is d'able,

$P = M \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix} M^{-1}$, some $M \Rightarrow P^n = M \begin{pmatrix} 1 & 0 \\ 0 & (0.3)^n \end{pmatrix} M^{-1}$

So as $n \to \infty$, P^n <u>stabilizes</u> to $M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1}$

$\rightsquigarrow$ Small comp $P^n \xrightarrow[n\to\infty]{} \begin{pmatrix} 5/7 & 5/7 \\ 2/7 & 2/7 \end{pmatrix}$ <u>exponentially fast</u>.

<u>note</u>: $\begin{pmatrix} 5/7 \\ 2/7 \end{pmatrix}$ is a 1-eigenvector of P.

$\Rightarrow$ 5/7 of days are sunny, 2/7 are cloudy!

Generally, a transition matrix P (stochastic) has all entries ~~pos.~~ non-neg., columns sum to 1, is square.

or equivalently, $P^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

But ~~P^T~~, P $\text{char}(P^T) = \text{char}(P)$

$\Rightarrow$ P has 1 as an eigenvalue.

Thm: If P is a transition matrix, then every eigenvalue λ has $|\lambda| \le 1$.

If some power of P has all positive entries, then only $\lambda = 1$ has abs. value 1, and the 1-eigenspace is 1-dimensional.

Then, the limit $\lim_{n\to\infty} P^n$ exists and has all columns equal to a steady state sol^n^ of the corr. Markov chain

In general, may not get diagonalizable matrices, but JCF is useful for taking powers too:

Say $P = MJM^{-1}$, $J = JCF(P)$. Then $P^n = MJ^nM^{-1}$,

$J^n = \begin{pmatrix} J_{m_1}^n(\lambda_1) & & \\ & \ddots & \\ & & J_{m_s}^n(\lambda_s) \end{pmatrix}$, and for blocks: $J_{m_i}^n(\lambda_i) = \ldots$

↑ take powers of blocks

and even better, on blocks, we have:

$$J_{m_i}^n(\lambda_i) = \begin{pmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \cdots & \binom{n}{m_i-1}\lambda_i^{n-m_i+1} \\ 0 & \lambda_i^n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{n}{2}\lambda_i^{n-2} \\ \vdots & 0 & \ddots & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} \\ 0 & \cdots & \cdots & 0 & \lambda_i^n \end{pmatrix}$$

size of block is $m_i \times m_i$, and $n \geq m_i - 1$.

These examples are important in lots of places, like Google page rank algorithm!