Computing invariant factors:

Fact: Given $A \in Mat_{n\times n}(F)$, consider $xI - A \in Mat_{n\times n}(F[x])$

Using elem. row/column ops,
one can put $xI - A$ in the form

$$xI - A \rightsquigarrow \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & a_1(x) & & \\ & & & & \ddots & \\ & & & & & a_m(x) \end{pmatrix},$$ Smith normal form,

where $a_i(x) \in F[x]$ are non-zero, monic, $\deg \geq 1$,
and $a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x)$.
Then the $a_1, \ldots, a_m$ are the invariant factors of A.
Further, if you keep track of these ops, you can explicitly write the matrix P s.t.

$P^{-1}AP$ is in RCF (see book/posted notes)

Example: $A := \begin{pmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{pmatrix} \rightarrow xI - A = \begin{pmatrix} x+1 & 2 & -6 \\ 1 & x & -3 \\ 1 & 1 & x-4 \end{pmatrix}$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 1 & x & -3 \\ x+1 & 2 & -6 \\ 1 & 1 & x-4 \end{pmatrix} \xrightarrow[\substack{R_2 - (x+1)R_1 \\ R_3 - R_1}]{} \begin{pmatrix} 1 & x & -3 \\ 0 & 2-x-x^2 & 3x-3 \\ 0 & 1-x & x-1 \end{pmatrix}$$

$$\xrightarrow[\substack{R_2 \leftrightarrow R_3 \\ -R_2}]{} \begin{pmatrix} 1 & x & -3 \\ 0 & x-1 & 1-x \\ 0 & -(x-2)(x-1) & 3x-3 \end{pmatrix} \xrightarrow[C_2 - x\cdot C_1]{} \begin{pmatrix} 1 & 0 & -3 \\ 0 & x-1 & 1-x \\ 0 & -(x-2)(x-1) & 3x-3 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3 + R_2\cdot(x+2) \\ -R_3}]{} \begin{pmatrix} 1 & 0 & -3 \\ 0 & \cancel{1-x}\ x-1 & 1-x \\ 0 & 0 & -(x-1)^2 \end{pmatrix} \xrightarrow[\substack{C_3 + 3C_1 \\ C_3 + C_2 \\ -R_3}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & (x-1)^2 \end{pmatrix}$$

$\rightarrow$ the invariant factors are $(x-1)$, $(x-1)^2$

with companion matrices (1), $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$\rightarrow RCF(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

Easier way: char poly $(A) = \det(xI - A)$

$$= \begin{vmatrix} x+1 & 2 & -6 \\ 1 & x & -3 \\ 1 & 1 & x-4 \end{vmatrix} = \begin{matrix} x+1 & 2 & -6 & x+1 & 2 \\ 1 & x & -3 & 1 & x \\ 1 & 1 & x-4 & 1 & 1 \end{matrix}$$

(Rule of Sarrus)

$= (x+1)x(x-4) - 6 - 6 + 6x + 3x + 3 - 2(x-4)$

$= (x-1)^3$ = prod. of inv. factors

Possible inv. factors: $x-1, x-1, x-1$ (identity matrix) min poly $= x-1$

$(x-1)(x-1)^2 \rightarrow$ min poly $= (x-1)^2$

$(x-1)^3 \rightarrow$ min poly $= (x-1)^3$

Check: $A - 1 \neq 0$ (clear)

$(A-1)^2 = 0 \Rightarrow$ min poly $= (x-1)^2$

$\Rightarrow$ inv. factors are $(x-1)(x-1)^2$