How to compute discriminants of general polynomials:

Def^n (Resultant) $p(x) = a_n x^n + \dots + a_0$; $q(x) = b_m x^m + \dots + b_0$.

Then the resultant of p and q is

$$\rho(p,q) := a_n^m b_m^n \prod_{i=1}^{m} \prod_{j=1}^{m} (\alpha_i - \beta_j),$$ where $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_m$ are the roots of p, q (resp.).

Fact: The resultant is the determinant of the Sylvester matrix:

$$\rho(p,q) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_0 & & \\ & & a_n & \cdots & \cdots & a_0 & \\ & & & \ddots & & & \\ & & & a_n & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & & & \\ & b_m & \cdots & b_0 & & & \\ & & b_m & \cdots & b_0 & & \\ & & & \ddots & & & \\ & & & b_m & \cdots & b_0 & \end{vmatrix}$$

blank entry means 0.

[This comes from solving for p, q to have a common root $\Leftrightarrow \exists$ polys a, b s.t. $a f = b g$, $\deg(a) \le m-1$, $\deg(b) \le n-1$ $\Leftrightarrow$ common divisor

and solving the coefficients of this eqs to get a system of linear equations. Then one expands this determinant carefully. This is in D–F Exercises 14.6. 29–32].

Thus, $D \neq 0$ $\operatorname{disc}(p) \neq 0 \Leftrightarrow$ no repeated roots $\Leftrightarrow \rho(p, p') \neq 0$.

Better, we can write $\boxed{D(p) = (-1)^{\frac{n(n-1)}{2}} a_n^{n-1} \rho(p, p').}$

All of this can be <u>assumed</u>.

<u>Ex</u>: Disc of cubics:

$f(x) = x^3 + bx + c, \quad f'(x) = 3x^2 + b$

$$\rho(f,f') = \begin{vmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 3 & 0 & b & 0 & 0 \\ 0 & 3 & 0 & b & 0 \\ 0 & 0 & 3 & 0 & b \end{vmatrix} \underset{\text{col 1}}{\overset{\text{Expand on}}{\Longrightarrow}} \begin{vmatrix} 1 & 0 & b & c \\ \cancel{0} & \cancel{b} & \cancel{0} & \cancel{0} \\ 3 & 0 & b & 0 \\ 0 & 3 & 0 & b \end{vmatrix} + 3 \begin{vmatrix} 0 & b & c & 0 \\ 1 & 0 & b & c \\ 3 & 0 & b & 0 \\ 0 & 3 & 0 & b \end{vmatrix}$$

$$= b \begin{vmatrix} 1 & b & c \\ 3 & b & 0 \\ 0 & 0 & b \end{vmatrix} - 3 \begin{vmatrix} b & c & 0 \\ \cancel{0} & \cancel{b} & \cancel{0} \\ 3 & 0 & b \end{vmatrix} + 9 \begin{vmatrix} b & c & 0 \\ 0 & b & c \\ 3 & 0 & b \end{vmatrix}$$

$$= b^2 \begin{vmatrix} 1 & b \\ 3 & b \end{vmatrix} - 3b \begin{vmatrix} b & 0 \\ 3 & b \end{vmatrix} + 9b \begin{vmatrix} b & c \\ 0 & b \end{vmatrix} - 9c \begin{vmatrix} 0 & c \\ 3 & b \end{vmatrix}$$

$$= b^2(-2b) - 3b(b^2 - 0) + 9b(b^2) - 9c(-3c)$$

$$= -2b^3 - 3b^3 + 9b^3 + 27c^2 = \boxed{4b^3 + 27c^2}.$$

<u>Ex</u>: (from page 643 of book): $f(x) = x^5 + 15x + 12$

$f'(x) = 5x^4 + 15$.

$$\rho(f,f') = \begin{vmatrix} 1 & 0 & 0 & 0 & 15 & 12 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 15 & 12 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 15 & 12 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 15 & 12 \\ 5 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 15 \end{vmatrix} = 259200000$$

$$= 2^{10} \cdot 3^4 \cdot 5^5. \quad = (-1)^{10}\operatorname{disc}(f) = \operatorname{disc}(f).$$

(So the "bad primes" are 2, 3, 5.)

For homework, feel free to just use a computer for this determinant, or even to ask a computer for discriminant.