HANDOUT: BURNSIDE'S LEMMA/PÓLYA ENUMERATION

In these notes, you will learn how to count configurations and labeling or colorings of geometric object up to symmetries. Doing so directly can quickly become a daunting combinatorial challenge. The calculations can often be shortened using a corollary of the Orbit-Stabilizer Theorem.

Follow the steps on this handout to see this result, an application of it, and a few fun problems to work through to test your ability in using this technique.

(1) Read through and follow the steps of the following theorem and its proof.

Theorem (Burnside's Lemma)**.** *Suppose that a finite group G acts on a finite set X. Define a function $f\colon G \to \mathbb{N}$ by letting $f(g)$ be the number of points of X fixed by g. Then the number of orbits of of this action is equal to*

$$\frac{1}{|G|}\sum_{g\in G} f(g).$$

In other words, the number of orbits is the average number of points fixed by elements of G.

Proof. We first rewrite the definition of being a fixed point under the action of an element g to obtain

$$\frac{1}{|G|}\sum_{g\in G} f(g) = \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{x\in X\\ g\cdot x=x}} 1.$$

Rearranging summations (for example, think of either as counting the number of pairs (g, x) of elements of G and X for which $g \cdot x = x$), this becomes

$$\frac{1}{|G|}\sum_{x\in X}\sum_{\substack{g\in G\\ g\cdot x=x}} 1.$$

Since the orbits of the action partition X, we can break this sum into a sum over orbits and then a sum over elements in a fixed orbit, and as $\sum_{\substack{g\in G\\ g\cdot x=x}} 1$ is the size of the stabilizer G_x, we obtain

$$\sum_{\text{orbits } \mathcal{O}} \frac{1}{|G|}\sum_{x\in\mathcal{O}} |G_x|.$$

By the Orbit-Stabilizer Theorem, the last quantity is the index of the orbit Gx in G, yielding

$$\sum_{\text{orbits } \mathcal{O}} \frac{1}{|G|} \sum_{x \in \mathcal{O}} \frac{|G|}{|Gx|} = \sum_{\text{orbits } \mathcal{O}} \sum_{x \in \mathcal{O}} \frac{1}{|\mathcal{O}|} = \sum_{\text{orbits } \mathcal{O}} 1,$$

which is the number of orbits, as claimed.

□

(2) As a first problem, follow the steps below to see how to count the number of ways to color the faces of a cube using 4 colors (where two colorings are considered the same if they are mapped to each other under a rotation). The group of rigid motions on a cube, as you perhaps found on the last homework when you studied the groups of rigid motions of a cube and an octahedron, is isomorphic to S_4. One way to see this is to consider the action of the symmetry group on pairs of opposite vertices and to check that it is a *simply transitive* action, which means that given any two x, y in the set acted on, there is a unique g with $g \cdot x = y$. Explicitly, we can classify and list these 24 symmetries. They are the following:
   (a) The identity transformation (1 total).
   (b) Rotations where the axis of rotation passes through two opposite faces. There are 3 possible axes of this form, and about each, there are 3 non-trivial rotations, of $90°, 180°$, and $270°$. There are thus 9 in this category.
   (c) There are 6 rotations of $180°$ about axes connecting the midpoints of opposite edges.
   (d) Finally, we have the rotations about the 4 axes which connect diagonally opposite vertices, which can be rotations of $120°$ or $240°$. There are thus 8 symmetries in this category.

   The group of rigid motions of the cube acts on the set of colored cubes, and the goal is to count the number of orbits. Now we count the number of fixed points under each symmetry:
   (a) The identity fixes everything, so it has 4^6 fixed points, as this is the number of colorings of the faces of the cube without restriction.
   (b) We break this down into two subcases: the rotations by $90°, 270°$, and the rotations by $180°$. In the first case, the two faces where the axis of rotation passes through are of course fixed, and the rotation acts as a 4-cycle on the other 4 faces. Thus, a coloring will be fixed iff it has the same color on the four faces which the axis of rotation doesn't pass through, and we can color the other two faces however we like, as the rotation acts as a 1-cycle on these two faces. This gives 4^3 possibilities for each rotation. As this illustrates, in general, the number of fixed points will be the number of colors, 4, to the power of the number of cycles in the corresponding permutation on the faces induced by the symmetry. Since there are 6 such rotations, this gives a total of $6 \cdot 4^3$ fixed points.

In the case where we rotate about one of these same axes of rotation a total of $180°$, the two faces which the axis passes through are still fixed, but now the two opposite pairs of the other four faces are interchanged. In other words, the induced permutation of the faces is a product of two transpositions, and there are 4 total cycles (including the 1 cycles). Thus, each such rotation has 4^4 fixed points, and, since there are 3 such rotations, we have to add $3 \cdot 4^4$ to our total count of fixed points so far.

(c) These types of rotations induce permutations on the faces of the cube which are a product of three transpositions, and so there are 4^3 fixed points for each of these rotations. Hence, these types of rotations contribute a total of $6 \cdot 4^3$ fixed points.

(d) These rotations correspond to two 3-cycles, and there are 8 of them, so these contribute a total of $8 \cdot 4^2$.

We can now use Burnside's Lemma to find the answer we are looking for. Adding all of these up and dividing by the order of the rigid symmetry group of the cube shows that the number of ways to paint a cube with 4 colors is

$$\frac{1}{24}\left(4^6 + 6 \cdot 4^3 + 3 \cdot 4^4 + 6 \cdot 4^3 + 8 \cdot 4^2\right) = \frac{1}{24} \cdot 5760 = 240.$$

(3) Note that in the above proof, we discovered that the number of fixed points of an action of a symmetry group acting on a geometric object adorned with k many colors is equal to k^c, where c is the number of cycles in the permutation of X corresponding to the symmetry (as a coloring will be fixed as long as all the elements of each cycle have the same color). This observation, when plugged into Burnside's Lemma, is also a famous result, and this method is known as *Pólya Enumeration*.

(4) Solve the following problem:

Benzene is a common organic compound, consisting of a ring of 6 carbon atoms C arranged in a hexagon and with an attached hydrogen atom H at each carbon atom. This ring is fully symmetric under rotations of the molecule and flipping it, and so the symmetries of D_{12} don't affect this molecule.

Things get more interesting when we allow other atoms besides hydrogen atoms to be attached to the carbon atoms. For example, suppose that we form new molecules by replacing some (maybe zero) of the hydrogens with either a CH_3 (methyl group) or an OH (hydroxide). How many different molecules can you create in this way, where two molecules are considered the same if they are rotations or flips of one another (i.e., if they are mapped to one another by an element of D_{12}?

(5) Solve the following problem:

Consider the usual tic-tac-toe board. How many way's are there to fill the board with 5 X's and 4 O's, where two fillings are considered the same if they are rotations or reflections of one another?