Distributions and inequalities for generalized partitions functions

Larry Rolen (joint with a number of collaborators)

The Partition function $p(n)$

Definition

A **partition** of an integer n is any nonincreasing sequence

$$\lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$$

of positive integers which sum to n.

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$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \quad \Longrightarrow \quad p(4) = 5.$$

Ramanujan's Legacy

Theorem (Hardy and Ramanujan)

We have that

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Theorem (Ramanujan)

For every n, we have that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Dyson's Rank

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Example

The ranks of the partitions of 4:

Partition	Largest Part	# Parts	Rank
4	4	1	$3 \equiv 3 \pmod 5$
$3+1$	3	2	$1 \equiv 1 \pmod 5$
$2+2$	2	2	$0 \equiv 0 \pmod 5$
$2+1+1$	2	3	$-1 \equiv 4 \pmod 5$
$1+1+1+1$	1	4	$-3 \equiv 2 \pmod 5$

Dyson's Conjecture

Conjecture (Dyson, 1944)

If $0 \leq a < b$, *then let*

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For every n *and every* a, *we have*

$$N(a, 5; 5n + 4) = p(5n + 4)/5,$$

$$N(a, 7; 7n + 5) = p(7n + 5)/7.$$

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$$\lim_{n \to +\infty} \frac{N(a, b; n)}{p(n)} = \frac{1}{b}.$$

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Question

For "large n", how are the actual ranks distributed?

Distribution of actual ranks

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The **rank polynomial** *for size n partitions is*

$$P(n;T) := \sum_{\lambda \vdash n} T^{\mathrm{rk}(\lambda)} = \sum_{m} N(m,n) T^{m}.$$

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Example

$$P(4;T) = T^{-3} + T^{-1} + 1 + T + T^{3},$$
$$P(5;T) = T^{-4} + T^{-2} + T^{-1} + 1 + T + T^{2} + T^{4}$$
$$P(6;T) = T^{-5} + T^{-3} + T^{-2} + 2T^{-1} + 1 + 2T + T^{2} + T^{3} + T^{5}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

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$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$P(50;T) = \frac{1}{T^{49}} + \frac{1}{T^{47}} + \cdots + \frac{9462}{T} + 9502 + 9462T + \cdots + T^{47} + T^{49}.$$

Distribution of the Number of Parts

Notation

The **"number of parts" polynomials** $P_{\#}(n;T)$ *are defined by*

$$\sum_{n=0}^{\infty} P_{\#}(n;T)q^n := \prod_{n=1}^{\infty} \frac{1}{(1-Tq^n)}.$$

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$$P_{\#}(6;T) = T + 3T^2 + 3T^3 + 2T^4 + T^5 + T^6$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$P_{\#}(15;T) = T + 7T^2 + 19T^3 + 27T^4 + 30T^5 + \cdots + 2T^{13} + T^{14} + T^{15}.$$

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(2) A limiting non-symmetric distribution for the # of parts?

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If k is a positive integer, then let

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If $C := \pi\sqrt{2/3}$ and $k_n := C^{-1}\sqrt{n}\log n + \sqrt{n}x$,

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THEOREM (ERDÖS AND LEHNER (1941))

If $C := \pi\sqrt{2/3}$ and $k_n := C^{-1}\sqrt{n}\log n + \sqrt{n}x$, then as a function in x we have

$$\lim_{n\to+\infty} \frac{p_{\leq k_n}(n)}{p(n)} = \exp\left(-\frac{2}{C}\cdot e^{-\frac{1}{2}Cx}\right).$$

Remarks

(1) **Normal order** *for the number of parts is*

$$\frac{\sqrt{n}\log n}{C} = \frac{\sqrt{3n}\log n}{\sqrt{2}\pi}.$$

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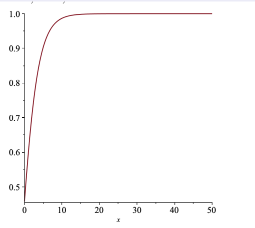

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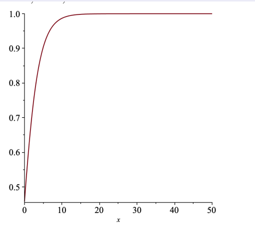

(3) For size n partitions, we use $k_n := C^{-1}\sqrt{n}\log n + \sqrt{n}x$.

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$$\text{Gumbel}(x) := \exp\left(-\frac{2}{C} \cdot e^{-\frac{1}{2}Cx}\right).$$

Partitions of $n = 750$

x	$\lfloor k_{750}(x) \rfloor$	$\delta_{k_{750}}(x)$	Gumbel(x)
0.5	84	$0.656\ldots$	$0.663\ldots$
1.0	98	$0.814\ldots$	$0.805\ldots$
1.5	111	$0.899\ldots$	$0.892\ldots$
2.0	125	$0.949\ldots$	$0.941\ldots$
2.5	139	$0.975\ldots$	$0.969\ldots$
3.0	152	$0.987\ldots$	$0.983\ldots$

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It is a nonsingular, irreducible, quasiprojective dimension $2n$ variety. The **Poincaré polynomial** (a.k.a. Betti number gen function) is

$$P\left(X^{[n]};T\right) := \sum_{j=0}^{2n-2} \dim\left(H_j\left(X^{[n]},\mathbb{Q}\right)\right) T^j.$$

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- Apply the Erdös-Lehner Theorem. □

Problem (Hausel & Rodriguez-Villegas)

Determine further limiting Betti distributions.

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(e.g. n point Hilbert schemes cut out by a torus action).

n point Hilbert schemes cut out by tori

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We have that

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This defines the **quasihomogeneous Hilbert scheme**

$$X^{[n]}_{\alpha,\beta} := ((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}.$$

Poincaré polynomials and distributions

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The **Poincaré polynomial** for $X_{\alpha,\beta}^{[n]} := ((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ is

$$P\left(X_{\alpha,\beta}^{[n]}; T\right) := \sum_{j=0}^{2\lfloor \frac{n}{\alpha+\beta} \rfloor} b_j(\alpha,\beta;n) T^j = \sum_{j=0}^{2\lfloor \frac{n}{\alpha+\beta} \rfloor} \dim\left(H_j\left(X_{\alpha,\beta}^{[n]}, \mathbb{Q}\right)\right) T^j.$$

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Definition

The **discrete measure** $d\mu_{\alpha,\beta}^{[n]}$ for $X_{\alpha,\beta}^{[n]}$ is

$$\Phi_n(\alpha, \beta; x) := \frac{1}{p(n)} \cdot \int_{-\infty}^{x} d\mu_{\alpha,\beta}^{[n]} = \frac{1}{p(n)} \cdot \sum_{j \leq x} b_j(\alpha, \beta; n).$$

Main Theorem

Theorem (Griffin, Ono, R., Tsai (2021))

If α *and* β *are relatively prime and* $\delta_n(\alpha, \beta) := \frac{\sqrt{6n} \cdot \log(n)}{\pi(\alpha+\beta)}$,

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(2) Again the limiting Betti distribution is Gumbel.

Where are the partitions?

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If $\alpha, \beta \in \mathbb{N}$ are relatively prime, then

$$G_{\alpha,\beta}(T;q) := \sum_{n=0}^{\infty} P\left(X_{\alpha,\beta}^{[n]};T\right) q^n = \frac{(q^{\alpha+\beta};q^{\alpha+\beta})_\infty}{(q;q)_\infty (T^2 q^{\alpha+\beta};q^{\alpha+\beta})_\infty}.$$

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What is the **cumulative distribution function** *for*

$$\frac{p_{\leq k}(A; n)}{p(n)} \, ?$$

Our partition result

Theorem (Griffin, Ono, R., Tsai (2021))

Suppose that $A \geq 2$ *is an integer. If* $C := \pi\sqrt{2/3}$ *and* $k_n = k_n(x) := \frac{1}{AC}\sqrt{n}\log n + \sqrt{n}x$,

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Remarks

(1) These are Gumbel distributions.

(2) The mean and variance of the limiting distribution are:

$$\text{Mean} := \frac{2}{AC}\left(\log\left(\frac{2}{AC}\right) + \gamma_{Euler}\right),$$

$$\text{Variance} := 1/A^2.$$

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Numerics when $A = 2$

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$$\text{Gumbel}(x) := \exp\left(-\frac{1}{C} \cdot e^{-Cx}\right).$$

Distribution of even parts for $n = 600$

x	$\lfloor k_{600}(x) \rfloor$	$\delta_{k_{600}}(x)$	Gumbel(x)
-0.1	28	$0.597\ldots$	$0.604\ldots$
0.0	30	$0.663\ldots$	$0.677\ldots$
0.1	32	$0.721\ldots$	$0.739\ldots$
0.2	35	$0.791\ldots$	$0.792\ldots$
0.3	37	$0.830\ldots$	$0.835\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.5	67	$0.994\ldots$	$0.992\ldots$
2.0	79	$0.998\ldots$	$0.998\ldots$

Asymptotics for $p_{\leq k}(A;n)$

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If $A \geq 2$, then as $n \to +\infty$ we have

$$p_{\leq k}(A;n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} n^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)n}},$$

$$p_{k}(A;n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} (n-Ak)^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)(n-Ak)}}.$$

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Proof.

- Need generating functions and that $p_k(A;n) = p_{\leq k}(A;n-Ak)$.

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- Ingham's Tauberian Theorem à la Bringmann et. al. □

Plane partitions

Definition

A *plane partition* of size n is an array of non-negative integers $\pi := (\pi_{i,j})$ for which $|\pi| := \sum_{i,j} \pi_{i,j} = n$, in which the rows and columns are weakly decreasing.

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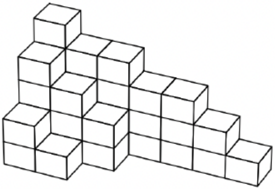
5	4	4	3	3	2	1
4	3	2				
2	1					

$\longrightarrow$



Generating function

- MacMahon:

$$f(x) = \sum_{n=0}^{\infty} \mathrm{PL}(n) x^n := \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^n}$$

$$= 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + \ldots.$$

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$$q\frac{d}{dq}\log(f(q)) = \sum_{m\geq 1} \frac{q^m(1+q^m)}{(1-q^m)^3} = \sum_{m\geq 1} \frac{m^2 q^m}{1-q^m}$$

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is a quantum modular "weight 3" Eisenstein series.

Partition inequalities

Theorem (De Salvo-Pak, Nicolas)

The partition function is eventually log-concave:

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Definition (Jensen)

For $a : \mathbb{N} \mapsto \mathbb{R}$ the **Jensen polynomial of deg.** d**, shift** n is

$$J_a^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} a(n+j) \cdot X^j.$$

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Table 1. Conjectured minimal values of $N(d)$

d	1	2	3	4	5	6	7	8	9
$N(d)$	1	25	94	206	381	610	908	1269	1701

Previous work

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A similar result is true where the coefficients of the Jensen polynomials arise from the Riemann Ξ-function.

Hermite Polynomials

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The **Hermite polynomials** $\{H_d(X) \ : \ d \geq 0\}$ are the orthogonal polynomials with respect to the measure $\mu(X) := e^{-X^2}$.

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Example

The first few Hermite polynomials

$$
\begin{aligned}
H_0(X) &= 1, \\
H_1(X) &= 2X, \\
H_2(X) &= 4X^2 - 2, \\
H_3(X) &= 8X^3 - 12X, \\
H_4(X) &= 16X^4 - 48X^2 + 12.
\end{aligned}
$$

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$$\log\left(\frac{a(n+j)}{a(n)}\right) = g_1(\varepsilon(n))j + g_2(\varepsilon(n))j^2 + o(g_2(\varepsilon(n))),$$

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where $g_2(\varepsilon(n)) = O(\varepsilon(n))$ is negative for large n.

Normalized Jensen Polynomials

Notation

If $a(n)$ has **appropriate growth**, *then let*

$$E(w) := e^{g_1(w)} \quad \text{and} \quad \delta(w) := 2\sqrt{-g_2(w)}.$$

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Definition

The **normalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{2^d}{\delta^d \cdot a(n)} \cdot J_a^{d,n}\left(E^{-1}\left(\delta X - 1\right)\right).$$

General Theorem

Theorem (Griffin, Ono, R, Zagier)

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For each degree $d \geq 1$ we have

$$\lim_{n \to +\infty} \widehat{J}_a^{d,n}(X) = H_d(X).$$

In particular, for each d, all but (possibly) finitely many $J_a^{d,n}(X)$ are hyperbolic.

Conjectures for plane partitions

Conjecture (Heim, Neuhauser, and Tröger)

The function $\mathrm{PL}(n)$ *is log-concave for* $n \geq 12$*. Moreover, analogues of the Chen-Jie-Wang Conjecture on the higher degree Turán inequalities hold.*

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Theorem (Ono, Pujahari, R. (2021))

The Heim-Neuhauser-Tröger Conjecture on the log-concavity of $\mathrm{PL}(n)$ *is true. Moreover, if* d *is a positive integer, then* $J_{\mathrm{PL}}^{d,n}(X)$ *is hyperbolic for all sufficiently large* n.

Work of Wright

$$A := \zeta(3) \approx 1.202056\ldots, \quad c := \zeta'(-1) \approx -0.16542\ldots.$$

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Theorem (Wright (1931))

As $n \to \infty$, we have

$$\mathrm{PL}(n) \sim \frac{(2^{25}A^7)^{\frac{1}{36}} e^c}{\sqrt{12\pi} \cdot n^{\frac{25}{36}}} \exp\left(\sqrt[3]{\frac{27An^2}{4}}\right).$$

A few required definitions

- We have the constants:

$$\frac{(1+y)^{2s+2m+\frac{13}{12}}}{(3+2y)^{(m+\frac{1}{2})}} =: \sum_{n=0}^{\infty} c_{s,m}(n)y^n, \qquad b_{s,m} := c_{s,m}(2m).$$

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- We also set $\alpha_s := \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}$,

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- It is convenient to set $N_n := \left(\frac{n}{2A}\right)^{\frac{1}{3}}$.

Threshold functions

- We need the thresholds:

$$n_r := \min \left\{ n \geq 1 \ : \ 0.056 \cdot \sum_{s=1}^{r+1} \left(\frac{s \cdot A^{\frac{1}{3}}}{2^{\frac{7}{6}} n^{\frac{1}{3}}} \right)^{2s} \left(\frac{\pi^2 n^{\frac{1}{3}}}{(2A)^{\frac{1}{3}} s} + 2 \right) < 1 \right\}$$

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$$\ell_r := \min\left\{ n \geq 1 \ : \ 2^{r+4} \pi^3 \alpha_{r+2} N_n^{-2r-4} + 5e^{-4.7N_n} < \frac{1}{2} \right\}.$$

Functions on a certain compact curve

- Wright performed a Circle Method analysis (+steepest descent) to analyze

$$J(n) \approx e^c \cdot \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N_n^{2s+\frac{13}{12}}},$$

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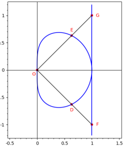

Error bound function

- First let $C_r := 2\max_{|z|=1}(|e^{-\sum_{s=1}^{r+1}\alpha_s z^s}|)$ and

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- Set $D_r := \frac{1}{(2r+4)!}\cdot\max\{\max\{|\chi_s^{(2r+4)}(t)|\}_{t\in\mathbb{R}}\}_{s=0}^{r+1}$ and

$$\mathcal{Z}_r(n) := e^c\left(D_r\cdot\Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r}e^{3AN_n^2}+0.64\cdot 2^{r+1}e^{2AN_n^2}\right)\sum_{s=0}^{r+1}\beta_sN_n^{-2s-\frac{13}{12}}.$$

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- Finally, we take

$$\widehat{E}_r^{\mathrm{maj}}(n) := \frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))e^{2AN_n^2}}{N_n\pi}+|\mathcal{Z}_r(n)|.$$

Effective version of Wright's asymptotic

Theorem (Pujahari, Ono, R. (2021))

If $r \in \mathbb{Z}^+$, *then for every integer* $n \geq \max(n_r, \ell_r, 87)$, *we have*

$$\mathrm{PL}(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1} \sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m} \Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n),$$

where $|E_r^{\mathrm{maj}}(n)| \leq \widehat{E}_r^{\mathrm{maj}}(n)$ *and*

$$|E^{\mathrm{min}}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right) n^2/(2A)^{\frac{2}{3}}\right).$$

Explicit versions for small values of r

- The thresholds ℓ_r and n_r are simple to compute.

EXPLICIT VERSIONS FOR SMALL VALUES OF r

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$$\begin{aligned}
v' &= tv^3/(v^3-1), \\
v'' &= v^3(1+3t^2v^2-2v^3+v^6)/(v^3-1)^3, \\
v^{(3)} &= 3tv^5(3+5t^2v^2-6v^3+4t^2v^5+3v^6)/(1-v^3)^5, \\
&\vdots \\
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$$\to \chi_1^{(6)}(t) = \frac{iv^{\frac{121}{12}}\sqrt{2v+1}}{859963392\pi(v^2+v+1)^{17}} \cdots (181387629768625 + \ldots + 2444688400v^{20}).$$

The case $r = 1$

For $n \geq 105$, the $r = 1$ case implies

$$\mathrm{PL}(n) = e^{3\cdot 2^{-\frac{2}{3}} A^{\frac{1}{3}} n^{\frac{2}{3}}} n^{-\frac{25}{36}} \Bigg(\frac{2^{\frac{25}{36}} e^{c} A^{\frac{7}{36}}}{\sqrt{12\pi}} - \frac{\sqrt{3}\cdot 2^{\frac{13}{36}} e^{c} (3A + 1385)}{25920 \sqrt{\pi} A^{\frac{5}{36}}} n^{-\frac{2}{3}} - \frac{\sqrt{3}\cdot 2^{\frac{1}{36}} e^{c} \left(1377A^2 - 370650A + 12525625\right)}{1567641600\sqrt{\pi} A^{\frac{17}{36}}} n^{-\frac{4}{3}} + E(n) \Bigg),$$

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n	$\mathrm{PL}(n)$	$E(n)$	$527n^{-\frac{5}{3}}$
100	$5.92\ldots \times 10^{16}$	$-1.18\ldots \times 10^{-7}$	$0.24\ldots$
200	$4.06\ldots \times 10^{27}$	$-3.00\ldots \times 10^{-8}$	$0.07\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	$2.91\ldots \times 10^{52}$	$-4.87\ldots \times 10^{-9}$	$0.01\ldots$

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$$\begin{aligned} \widehat{\mathrm{PL}}_2(n) := e^{3AN_n^2} n^{-\frac{25}{36}} &(f_{0,0}(n) + f_{1,0}(n) + f_{0,1}(n) + f_{1,1}(n) + f_{2,0}(n) + f_{0,2}(n) \\ &+ f_{3,0}(n) + f_{0,3}(n) + f_{2,1}(n) + f_{1,2}(n)) \\ &\approx e^{c+3AN_n^2} n^{-\frac{25}{36}} (0.23 - 0.056n^{-\frac{2}{3}} - 0.006n^{-\frac{4}{3}} - 0.001n^{-2}). \end{aligned}$$

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$$n \geq 87 \implies |\mathrm{PL}(n) - \widehat{\mathrm{PL}}_2(n)| \leq \mathcal{E}_2(n) := 227 e^{3AN_n^2} n^{-\frac{109}{36}} + e^{\left(3A - \frac{2}{5}\right) N_n^2}.$$

Numerics for $r = 2$

n	$\widehat{\mathrm{PL}}_2(n) - \mathcal{E}_2(n)$	$\mathrm{PL}(n)$	$\widehat{\mathrm{PL}}_2(n) + \mathcal{E}_2(n)$
100	$5.932\ldots \times 10^{15}$	$5.920\ldots \times 10^{16}$	$1.124\ldots \times 10^{17}$
200	$3.706\ldots \times 10^{27}$	$4.066\ldots \times 10^{27}$	$4.426\ldots \times 10^{27}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	$2.913\ldots \times 10^{52}$	$2.915\ldots \times 10^{52}$	$2.917\ldots \times 10^{52}$
1000	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$
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1000	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$

- This is good enough to show that $\mathrm{PL}(n)$ is log-concave for all $n \geq 8820$.