Conjectures of Andrews on partition-theoretic q-series

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$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} =: \sum_{n\geq 0} S(n)q^n$$

$$= 1+q-q^2+2q^3-2q^4+q^5+q^7-2q^8+2q^{10}-q^{12}-2q^{13}+O(q^{14}).$$

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- No $|S(n)|$ for $n \leq 1600$ is ≥ 4, but term can exceed 10^{13}.

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- Generating function version: indefinite theta function

$$q\sigma(q^{24}) = \sum_{a>6|b|} \left(\frac{12}{a}\right) (-1)^b q^{a^2-24b^2}.$$

Even deeper structure

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- Zagier: These are *period functions* of the Maass waveform, and give *quantum modular forms.*

Other functions

- Another function from the Lost Notebook:

$$v_1(q) := \sum_{n \geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q^2; q^2)_n} =: \sum_{n \geq 0} V_1(n) q^n.$$

Other conjectures

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For almost all n, $V_1(n)$, $V_1(n+1)$, $V_1(n+2)$ *and* $V_1(n+3)$ *are two positive and two negative numbers.*

Data

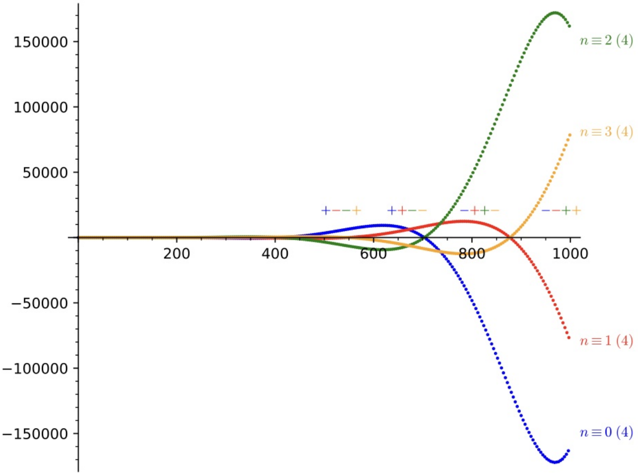

FIGURE 1. $V_1(n)$ for $n = 1, \ldots, 1000$

Main Result

Theorem (Folsom, Males, R., Storzer (2023))

The twos conjectures of Andrews above are true.

Asymptotics near roots of unity $\zeta = e(\alpha)$ of order m

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1. *If* $4 \nmid m$, *then* $v_1(\zeta e^{-z}) = O(1)$ *as* $z \to 0$.

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2. *If* $4 | m$, *then as* $z \to 0$,

$$v_1(\zeta e^{-z}) = e^{\frac{16V}{zm^2}} \sqrt{\frac{2\pi i}{z}} \left(\gamma^{+}_{(\alpha)} + O(|z|)\right) + e^{\frac{-16V}{zm^2}} \sqrt{\frac{2\pi i}{-z}} \left(\gamma^{-}_{(\alpha)} + O(|z|)\right).$$

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Here using Bloch-Wigner dilogarithm: $V = \mathrm{D}(e(1/6))\frac{i}{8}$, *gamma numbers e.g.:*

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Here using Bloch-Wigner dilogarithm: $V = \mathrm{D}(e(1/6))\frac{i}{8}$, *gamma numbers e.g.:*$\gamma^+ := \gamma^{+}_{(1/4)} = \gamma^{-}_{(3/4)} = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}}$ *and* $\gamma^- := \gamma^{-}_{(1/4)} = \gamma^{+}_{(3/4)} = \frac{1}{2\sqrt[4]{3(2+\sqrt{3})}}$.

"Plugging into" Wright's Circle Method

Theorem (Folsom, Males, R., Storzer (2023))

As $n \to \infty$ we have

$$V_1(n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{e^{\sqrt{2|V|n}}}{\sqrt{n}} (\gamma^+ + (-1)^n \gamma^-) \times \left(\cos(\sqrt{2|V|n}) - (-1)^n \sin(\sqrt{2|V|n}) \right) \left(1 + O\left(n^{-\frac{1}{2}} \right) \right) + O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{|V|n}{2}}} \right).$$

Sign pattern explanation (Conjecture 2)

- Asymptotics for $V_1(n)$ reduce us to study signs of

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- Main term "wins" if not "very" close to root of $\cos(x) \pm \sin(x)$. Erdös-Turán$+\delta \implies$ fails $\ll \sqrt{X}$ of time.

Other conjectures for $V_1(n)$

Conjecture (Andrews)

For $n \geq 5$ *there is an infinite sequence* $N_5 = 293, N_6 = 410, N_7 = 545, N_8 = 702, \ldots, N_n \geq 10n^2, \ldots$ *such that* $V_1(N_n), V_1(N_n + 1), V_1(N_n + 2)$ *all have the same sign.*

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Conjecture (Andrews)

The numbers $|V_1(N_n)|, |V_1(N_n + 1)|, |V_1(N_n + 2)|$ contain a local minimum of the sequence $|V_1(j)|$.

Diophantine Analysis

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- Up to 5 million coefficients (715 sign pattern failures), all of our integers are within 2 of the conjectural infinite sequence.
- Milnor $\implies$ $|V| = \frac{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}{16\pi^2}$.

More on these sorts of constants

- Siegel-Klingen: Used Hilbert modular forms to show that $\zeta_K(2n) \in \sqrt{|\operatorname{disc}(K)|}\pi^{2kN}\mathbb{Q}$ for $n \in \mathbb{N}$, K totally real.

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- Zagier: $\zeta_K(2)$ for arbitrary number fields represented via powers of π, $\sqrt{\operatorname{disc}(K)}$ and integrals of the shape

$$A(x) = \int_0^x \frac{1}{1+t^2} \log \frac{4}{1+t^2} dt.$$

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- Is this a hint of a modular object involving $\mathbb{Q}(\sqrt{-3})$???

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- Andrews' intuition and our results imply that there could be deep modular arithmetic lurking. Modular forms tend to leave their "fingerprints."
- We prove, or at least "explain" modulo hard irrationality questions, the conjectures of Andrews on V_1. There are additional functions with similar conjectures in his paper!