Inequalities for the partition function and other combinatorial sequences

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Partitions

Integer partitions

Definition

An **integer partition** *of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that*

$$\lambda_1 + \ldots + \lambda_k = n.$$

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$$13 = \textit{Eleven Plus Two} = \textit{Twelve Plus One}.$$

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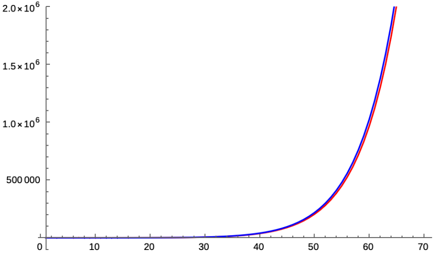
[Credit:Jon Perry (June 2011)]



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- This can be extended to an **exact formula** of Rademacher:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k \geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

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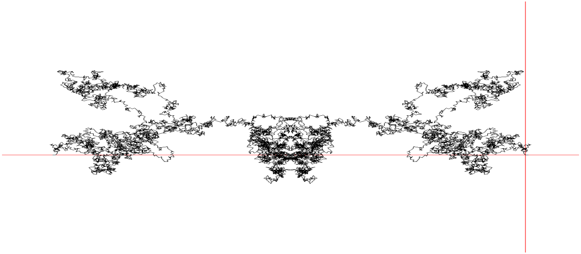

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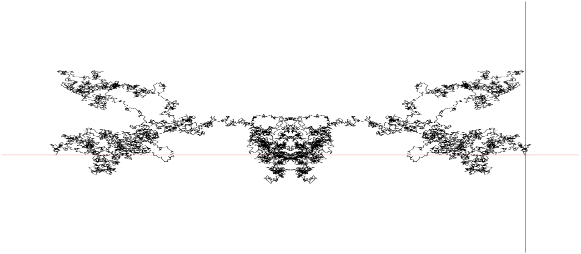

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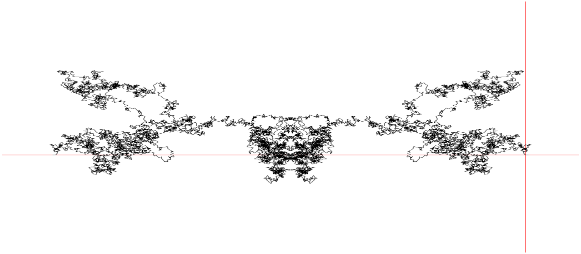

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$$p(n)^2 \geq p(n-1)p(n+1) \qquad (n > 25).$$

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Theorem (Ramanujan's Congruences 1919; Hardy–Ramanujan)

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- Congruences exist for other primes, but they look like this:

$$p(107^4 \cdot 31k + 30064597) \equiv 0 \pmod{31} \qquad \text{Ono, 2000.}$$

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Theorem (Chen–Jia–Wang)

For $n \geq 95$, $p(n)$ *satisfies the higher Turán inequality:*

$$4\left(p^2(n) - p(n-1)p(n+1)\right)\left(p^2(n+1) - p(n)p(n+2)\right)$$

$$-\left(p(n)p(n+1) - p(n-1)p(n+2)\right)^2 \geq 0.$$

Chen's reformulation

Definition

The *Jensen polynomial* of degree d and shift n associated to $\{a(n)\}_{n\geq 0}$ is

$$J_a^{d,n}(x) := \sum_{j=0}^{d} \binom{d}{j} a(n+j)x^j.$$

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- The higher Turán inequalities hold iff $J_a^{3,n}(x)$ have real roots.
- The Jensen–Pólya program for the Riemann Hypothesis begins with the fact that the RH is equivalent to real-rootedness of all Jensen polynomials built out of Taylor coefficients for the Riemann ξ-function.

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Conjecture (Chen–Jia–Wang)

Fix $d \geq 2$. Then there exists a constant N_d for which $J_p^{d,n}(x)$ has real roots for all $n \geq N_d$.

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Remark

Known bounds for N_d appear very far from the truth!

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5. These polynomials numerically stabilize! For example, the "renormalized" version at $d = 4$, $n = 10^{10}$ is $1.000\dots x^4 + 0.004\dots x^3 - 1.101\dots x^2 - 0.004\dots x + 0.101\dots$.

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6. Zagier noticed: These look like *Hermite polynomials!* Here, if we let $x \mapsto x/\sqrt{-1/2\sqrt{6}+3/2}$, multiply by $3920/33$, we get $16.000\dots x^4 + 0.109\dots x^3 - 47.999\dots x^2 - 0.298\dots x + 11.999\dots \approx 16x^4 - 48x^2 + 12 = H_4(x)$.

Work with Griffin, Ono, and Zagier

Theorem (Griffin–Ono–R.–Zagier, 2019)

For "suitable" sequences a_n such as $p(n)$ or the Taylor coefficients of the Riemann ξ-function, there are suitable sequences $\alpha_n, \beta_n, \gamma_n$ such that for any fixed $d \geq 2$,

$$\lim_{n\to\infty} \gamma_n \cdot J_a^{d,n}(\alpha_n x + \beta_n) \to H_d(x),$$

where $H_d(x)$ is the d-th Hermite polynomial.

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Corollary (Griffin–Ono–R.–Zagier, 2019)

The conjecture of Chen–Jia–Wang is true.

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Question

Are there partial converses of the theorem of Griffin–Ono–R.–Zagier one can establish?

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- We aim to study positivity/asymptotics of differences:

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- Setting $u_1 := 1$ and $u_T := 2$ if $T \geq 2$, we also require

$$D_t := \sum_{\ell=1}^{2} (-1)^{\ell+1} \sum_{m=1}^{M_\ell} \rho_\ell(m) \left(\sum_{k=1}^{T} s_\ell^{u_T}(k,m) \right)^t.$$

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3. We have $\lim_{n\to\infty} A(n) = 0$.
4. We have $D_t = 0$ for $0 \leq t \leq N-1$, $D_N \neq 0$ for some $N \in \mathbb{N}$.

Main result

Theorem (Banerjee–Bringmann–R., 2026+)

Under the assumptions above, set

$$\mathcal{M}(n) := \begin{cases} -\delta^2(n) & \text{if } T \geq 2, \\ A(n) & \text{if } \ T = 1. \end{cases}$$

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Then we have

$$I_{\rho,\mathrm{s}}^{[\mathrm{M}]}(a)(n) \sim \frac{D_N}{N!}\mathcal{M}^N(n) \quad (as\ n \to \infty)\,.$$

Basic Examples

1. Shifting $n \mapsto n+1$, taking $T=2, M_1 = M_2 = 1$, $\rho_1(1) = \rho_2(1) = 1$, $s_1(1,1) = s_1(2,1) = 1$, $s_2(1,1) = 0$, $s_2(2,1) = 2$, and $N = 1$ gives log-concavity.

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2. Taking $T = 4$, $M_1 = 2$, $M_2 = 3$, $\rho_1(1) = 3$, $\rho_1(2) = 6$, $\rho_2(1) = \rho_2(2) = 4$, $\rho_2(3) = 1$, $s_1(1,1) = s_1(2,1) = 0$, $s_1(3,1) = s_1(4,1) = 1$, $s_1(1,2) = -1$, $s_1(2,2) = 0$, $s_1(3,2) = 1$, $s_1(4,2) = 2$, $s_2(1,1) = s_2(2,1) = s_2(3,1) = 0$, $s_2(4,1) = 2$, $s_2(1,2) = -1$, $s_2(2,2) = s_2(3,2) = s_2(4,2) = 1$, $s_2(1,3) = s_2(2,3) = -1$, $s_2(3,3) = s_2(4,3) = 2$, and $N = 3$ gives the higher Turán inequality

Basic Examples

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2. Taking $T = 4$, $M_1 = 2$, $M_2 = 3$, $\rho_1(1) = 3$, $\rho_1(2) = 6$, $\rho_2(1) = \rho_2(2) = 4$, $\rho_2(3) = 1$, $s_1(1,1) = s_1(2,1) = 0$, $s_1(3,1) = s_1(4,1) = 1$, $s_1(1,2) = -1$, $s_1(2,2) = 0$, $s_1(3,2) = 1$, $s_1(4,2) = 2$, $s_2(1,1) = s_2(2,1) = s_2(3,1) = 0$, $s_2(4,1) = 2$, $s_2(1,2) = -1$, $s_2(2,2) = s_2(3,2) = s_2(4,2) = 1$, $s_2(1,3) = s_2(2,3) = -1$, $s_2(3,3) = s_2(4,3) = 2$, and $N = 3$ gives the higher Turán inequality
3. Moral: If you have a polynomial inequality you want to study, "write me a little email" and I will compute the parameters of our theorem that apply to your situation.

Sample applications

Conjecture (Peng–Zhang–Zhong)

For large n, the k-regular overpartition function, with gen. function

$$P_k(q) := \sum_{n\geq 0} \overline{p}_k(n)q^n = \frac{(-q)_\infty \left(q^k;q^k\right)_\infty}{(q)_\infty \left(-q^k;q^k\right)_\infty},$$

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Conjecture (Dong–Ji–Jia)

For Andrews–Paule's broken k-diamond partitions with gen. function $B_k(q) := \sum_{n \geq 0} b_k(n) q^n = \frac{(-q)_\infty}{(q)_\infty^2 \left(-q^{2k+1}; q^{2k+1}\right)_\infty}$, *the polynomial* $J_{b_k}^{d,n}$ *has real roots for fixed* $k \geq 3$, $d \geq 2$, *and* $n \gg 1$.

Sample applications (2)

Conjecture (Liu–Zhang)

For $n \gg 1$, the k-regular partition and overpartition functions (no parts divisible by k) satisfy the Briggs inequality:

$$a^2(n)\left(a^2(n) - a(n-1)a(n+1)\right) >$$

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Conjecture (Banerjee for $p(n)$, Mukherjee for $p_{24}(n)$)

Consider the iterated Laguerre operator
$L_r(a)(n) := \frac{1}{2}\sum_{m=0}^{2r}(-1)^{m+r}\binom{2r}{m}a(n+m)a(n+2r-m)$.

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$L_r(p)(n) \sim \frac{(2r-1)!!\pi^r}{2^{\frac{r}{2}+5}3^{\frac{r}{2}+1}} \frac{e^{2\pi\sqrt{\frac{2}{3}n}}}{n^{\frac{3r}{2}+2}}$.

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Sample applications (3)

Conjecture (Banerjee)

Let $\mathcal{L}^{[1]}(a)(n) := \{a^{[1]}(n)\}_{n\geq 0}$ *with*

$$a^{[1]}(0) := a^2(0) \ \ \text{and} \ \ a^{[1]}(n) := a^2(n) - a(n+1)a(n-1), \ \ \text{for } n \geq 1.$$

Recursively define $\mathcal{L}^{[r]} := \mathcal{L}^{[1]}\left(\mathcal{L}^{[r-1]}(a)\right)(n)$.

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Consider the Toeplitz determinants
$\mathrm{T}_a^{[r]}(n) := \det(a(n-k+\ell)_{1\leq k,\ell\leq r})$.

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Consider the Toeplitz determinants
$\mathrm{T}_a^{[r]}(n) := \det(a(n-k+\ell)_{1\leq k,\ell\leq r})$. *Then* $\mathrm{T}_p^{[r]}(n) > 0$ *for* $n \gg 1$.

Some theorems resulting from our framework

Theorem (Banerjee–Bringmann–R., 2026+)

The above conjectures are true.

Thank you!!!