TUTORIAL 9 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Evaluate $\int_0^1 \int_x^{2x} \int_{xy-1}^{x+y} x^2 dz dy dx$.
Solution: We have

$$\int_0^1 \int_x^{2x} \left(\int_{xy-1}^{x+y} x^2 dz \right) dy dx$$
$$= \int_0^1 x^2 \int_x^{2x} (x + y - xy + 1) dy dx$$
$$= \int_0^1 x^2 \left[y(x+1) + \frac{y^2(1-x)}{2} \right]_{y=x}^{y=2x} dx$$
$$= \int_0^1 x^2 \left(2x^2 + 2x + 2x^2 - 2x^3 - x^2 - x - \frac{x^2}{2} + \frac{x^3}{2} \right) dx.$$
$$= \int_0^1 \left(-\frac{3x^5}{2} + \frac{5x^4}{2} + x^3 \right) dx$$
$$= \left[-\frac{x^6}{4} + \frac{x^5}{2} + \frac{x^4}{4} \right]_{x=0}^1 = -\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2}.$$

(2) Use spherical coordinates to evaluate $\iiint_R x dV$ when R is the piece lying in the first octant of the unit ball $x^2 + y^2 + z^2 \leq 1$ centered at the origin. (Recall that $\sin^2 \vartheta = \frac{1-\cos(2\vartheta)}{2}$.)
Solution: This hemisphere corresponds to the range with $0 \leq \rho \leq 1$, $0 \leq \varphi \leq \frac{\pi}{2}$, $0 \leq \vartheta \leq \frac{\pi}{2}$. Moreover, since $x = \rho \sin \varphi \cos \vartheta$, and $dV = \rho^2 \sin \varphi d\rho d\varphi d\vartheta$, we

find that our integral equals

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\sin^2\varphi\cos\vartheta d\rho d\varphi d\vartheta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\vartheta d\varphi d\vartheta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\cos\vartheta\int_0^{\frac{\pi}{2}}\frac{1-\cos(2\varphi)}{2}d\varphi d\vartheta \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}}\cos\vartheta\left[\varphi-\frac{\sin(2\varphi)}{2}\right]_{\varphi=0}^{\frac{\pi}{2}}d\vartheta \\
&= \frac{\pi}{16}\int_0^{\frac{\pi}{2}}\cos\vartheta d\vartheta = \frac{\pi}{16}.
\end{aligned}$$

(3) Use a change of variables to find $\iint_R (y^2-x^2)^4 dA$ where R is the trapezoid with vertices at $(0,1)$, $(1,0)$, $(2,0)$, and $(0,2)$. (Hint: Make a change of variables which transforms two of the sides of the trapezoid to be on lines of the form $u=a$ and $u=b$, and to find a suitable second parameter v, make a choice which makes the integrand as nice as possible.)

Solution: The sides of the trapezoid are pieces of lines having equations given by $x=0$ (left side), $y=0$ (bottom side) $y=1-x$ (from $(0,1)$ to $(1,0)$), and $y=2-x$ for the remaining side. Two of the opposite sides have equations given by $x+y=1$, $x+y=2$, so this suggests that we can set $u=x+y$ as one of our variables. The function inside the fourth power to be integrated is $y^2-x^2=(x+y)(y-x)$, so this suggests that we make $v=x-y$ so that the integrand becomes $(-uv)^4=u^4v^4$. Solving for x and y, we obtain $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$. Now we need to find the corresponding region of integration in the u-v plane. We plug in to find the u,v coordinates of each of the four vertices above. Specifically, we find that the points $(0,1)$, $(1,0)$, $(2,0)$, and $(0,2)$ in x-y are the images of the points $(1,-1)$, $(1,1)$, $(2,2)$, and $(2,-2)$, respectively in u-v. The transformation also clearly takes lines to lines (you can also just solve $x=(u+v)/2=0 \implies v=-u$, $y=(u-v)/2=0 \implies v=u$), and takes two sides of the trapezoid to the lines $u=1$, $u=2$, as we also saw above. So, the new region is also a trapezoid, but now with two sides parallel to the v-axis, and with the other two sides lying on lines with the equations $v=u$ and $v=-u$. Thus, this is a type I region, with u ranging from 1 to 2 and where for fixed u, v ranges from $-u$ to u. The Jacobian of this transformation is

$$J(u,v)=\det\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=-\frac{1}{2},$$

which has absolute value 1/2. Thus, the integral is equal to

$$\frac{1}{2}\int_1^2 u^4\left(\int_{-u}^{u} v^4 dv\right)du = \frac{1}{5}\int_1^2 u^9 du = \frac{2^{10}-1}{50} = \frac{1023}{50}.$$

Advanced Problem:

Consider the n-variable function $e^{-\frac{1}{2}\left(x_1^2+x_2^2+\dots x_n^2\right)}$. Express $\int\cdots\int_{\mathbb{R}^n} e^{-\frac{1}{2}\left(x_1^2+x_2^2+\dots x_n^2\right)}dV$, where dV is an n-dimensional volume element, as a product of integrals to find its value. Additionally use the very important technique of differentiating under the integration sign, which states that for "nice" functions $f(x,t)$ we have

$$\frac{d}{dx}\int_a^b f(x,t)dx = \int_a^b \frac{\partial f}{\partial x}(x,t)dt,$$

to evaluate the integral

$$\int_{-\infty}^{\infty} x^n e^{-x^2}$$

for any positive integer n. (Hint: take the Gaussian integral identity $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ and insert a parameter t.)

Solution: By Fubini's theorem in n dimensions, we have

$$\begin{aligned}
&\int\cdots\int_{\mathbb{R}^n} e^{-\frac{1}{2}\left(x_1^2+x_2^2+\dots x_n^2\right)}dV \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\cdots e^{-\frac{x_n^2}{2}}dx_1\dots dx_n \\
&= \left(\int_{-\infty}^{\infty} e^{-\frac{x_1^2}{2}}dx_1\right)\cdots\left(\int_{-\infty}^{\infty} e^{-\frac{x_n^2}{2}}dx_n\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx\right)^n = (2\pi)^{\frac{n}{2}}
\end{aligned}$$

(applying a change of variables to the formula $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$). This useful formula can be used in a number of applications, for example in computing the (hyper)-volume of any n-dimensional sphere.

For the second integral, note that by a change of variables, $\int_{-\infty}^{\infty} e^{-tx^2}dx = \sqrt{\frac{\pi}{t}}$ (similar to, and motivated by, how the last change of variables worked). Now differentiate under the integral sign to find that

$$\frac{d}{dt}\int_{-\infty}^{\infty} e^{-tx^2}dx = -\frac{1}{2}\sqrt{\frac{\pi}{t^3}} = -\int_{-\infty}^{\infty} x^2 e^{-tx^2}dx.$$

Evaluating this at $t=1$ shows that $\int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$. Differentiating the equation above under the integration sign again shows that

$$\frac{3}{4}\sqrt{\frac{\pi}{t^5}} = \int_{-\infty}^{\infty} x^4 e^{-tx^2}dx,$$

and so

$$\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \frac{3}{4}\sqrt{\pi}.$$

Continuing in this way and observing the pattern (which can be proven using induction) shows that

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

for any positive integer n. This covers the even powers of x. For any odd number n, the integral $\int_{-\infty}^{\infty} x^n e^{-x^2}$ is zero, since the integrand is an odd function. Putting these two pieces together with the formula from last week's advanced problem, we find that for integers $n \geq 0$, we have

$$\int_{-\infty}^{\infty} x^n e^{-x^2} dx = \begin{cases} \Gamma\left(\frac{n+1}{2}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Since the integral we computed looks so similar to the one we computed in the Gamma function in last week's advanced problem, it is worth noting that you could derive this formula directly from last week's by applying a u-substitution to the integral defining the Gamma function. In fact, we used the reverse substitution in one step last time. However, this gives a different proof and was a good motivation for learning the very useful, common trick of differentiating under the integral sign.