TUTORIAL 8 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^5) y dx dy$$

by switching the order of integration. That is, write this as a double integral over a region R in the plane (sketch a picture), rewrite this as in integral with respect to $dydx$, and evaluate.

Solution: Since we don't know how to find antiderivative for $\sin(x^5)$, we switch the order of integration as suggested. The integral is equal to

$$\iint_R \sin(x^5) y dA,$$

where R is the region bounded by the parabola $y = x^2$ (solve $x = \sqrt{y}$ for y), the line $x = 1$, and the x-axis. Integrating this over y first, this is equal to

$$\int_0^1 \int_0^{x^2} \sin(x^5) y dy dx = \frac{1}{2} \int_0^1 \sin(x^5) x^4 dx = \frac{1}{10} \left[-\cos(x^5) \right]_{x=0}^1 = \frac{1 - \cos(1)}{10}.$$

(2) Evaluate

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{dydx}{\sqrt{x^2+y^2}}$$

by first switching to polar coordinates.

Solution: We first want to express this integral as an integral

$$\iint_R \frac{dA}{\sqrt{x^2+y^2}}$$

for some region R. The integration region, reading off from the double integral above, is bounded by the lines $x = 0$ and $x = 1$, on the left and right, and on the top and bottom by $y = x$ and the top arc of the circle $x^2 + y^2 = 2$ (note that the line $x = 1$, the circle, and the line $y = x$ all meet at the same point $(1, 1)$). Thus, R in this case is just a sector of a circle of radius $\sqrt{2}$ between the rays $\vartheta = \pi/4$ and $\vartheta = \pi/2$. Thus, recalling that $dA = r dr d\vartheta$ and $x^2 + y^2 = r^2$, the integral is equal to

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} dr d\vartheta = \frac{\pi\sqrt{2}}{4}.$$

(3) Show that the parametric surface

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = u^2 \end{cases}$$

with $1 \leq u \leq 4$ and $0 \leq v \leq \frac{\pi}{2}$ is a piece of the paraboloid $z = x^2 + y^2$. Find the surface area of this piece.

Solution: For any point on the surface, we have that $z = u^2 = u^2(\cos^2\vartheta + \sin^2\vartheta) = x^2 + y^2$. This is the graph of the vector-valued function $r(u, v) = (u\cos v, u\sin v, u^2)$. We first compute its partial derivatives:

$$r_u = (\cos v, \sin v, 2u),$$

$$r_v = (-u\sin v, u\cos v, 0).$$

The cross product is $r_u \times r_v = (-2u^2\cos v, -2u^2\sin v, u(\cos^2 v + \sin^2 v)) = u(-2u\cos v, -2u\sin v, 1)$, which has norm (note that $u > 0$ in our range)

$$u\sqrt{4u^2(\sin^2 v + \cos^2 v) + 1} = u\sqrt{4u^2 + 1}.$$

Thus, the surface area is the integral of this over the corresponding region in the u-v plane:

$$\iint_R |r_u \times r_v| dA = \int_0^{\frac{\pi}{2}} \int_1^4 u\sqrt{4u^2+1}dudv.$$

This is

$$\frac{1}{12}\int_0^{\frac{\pi}{2}} \left[(4u^2+1)^{\frac{3}{2}}\right]_{u=1}^4 dv = \frac{\pi}{24}(65^{\frac{3}{2}} - 5^{\frac{3}{2}}).$$

Advanced Problem:

The very important *Gamma function* is defined for positive real numbers as the integral

$$\int_0^\infty x^{z-1}e^{-x}dx.$$

Show that this function evaluates to $\Gamma(n) = (n-1)!$ at positive integers n. (Hint: use integration by parts to relate the values of $\Gamma(z)$ and $\Gamma(z+1)$.) The values at other places are interesting as well. Use a u-substitution to evaluate $\Gamma(1/2)$. Put together what you learned in the last two parts to find $\Gamma(n/2)$ for any positive integer n.

Solution: Firstly, we find that $\Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{a\to\infty}(1 - e^{-a}) = 1$. Thus, $\Gamma(1) = 1 = (1-1)!$, as claimed. Now, for general $z > 0$, using integration by parts shows that

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}dx = \lim_{a\to\infty}\left[-x^z e^{-x}\right]_{x=0}^a + \int zx^{z-1}e^{-x}dx = z\Gamma(z)$$

(here we used the fact that e^{-x} decays faster than any power of x like x^z). Thus,

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)\\ &= \cdots = (n-1)(n-2)\cdots 1\cdot\Gamma(1) = (n-1)(n-2)\cdots 2\cdot 1 = (n-1)!.\end{aligned}$$

At 1/2, we have to evaluate

$$\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx.$$

Using the substitution $x = u^2$ turns this into ($dx = 2udu$, so $dx/x^{\frac{1}{2}} = 2du$)

$$2\int_0^\infty e^{-u^2}du,$$

which by symmetry is $\int_{-\infty}^{\infty} e^{-u^2}du$. But this is an integral we saw how to handle using polar coordinates in class, so we have found $\Gamma(1/2) = \sqrt{\pi}$. We can use the equation $\Gamma(z+1) = z\Gamma(z)$ to evaluate Γ at all positive half-integers from this value. For example, $\Gamma(3/2) = \frac{1}{2}\cdot\Gamma(1/2) = \sqrt{\pi}/2$, $\Gamma(5/2) = \frac{3}{2}\cdot\Gamma(3/2) = \frac{3}{4}\cdot\sqrt{\pi}$. By noting the pattern from a few examples we can observe (or prove, using induction) that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

where $n!!$ denotes the double factorial $n(n-2)(n-4)\cdots\varepsilon$, with $\varepsilon = 2, 1$ depending on whether n is even or odd, respectively. Thus, we have computed $\Gamma(n/2)$ for all integers n (i.e., in both the cases when n is even or odd).