TUTORIAL 6 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Consider the surface S given by the equation $x^3y - xyz = 10$.
  (a) Find the equation for the tangent plane to this surface at $P = (2, 1, -1)$.
  (b) Find parametric equations for the tangent line of the curve of intersection of the surface S with the plane $z = -1$ at the same point $P = (2, 1, -1)$.

Solution:

a). The surface is the set of points where $F(x, y, z) = 0$, with $F(x, y, z) = x^3y - xyz - 10$. We then compute the gradient in general:

$$\nabla F = (F_x, F_y, F_z) = (3x^2y - yz, x^3 - xz, -xy)$$

and at P:

$$\nabla F(2, 1, -1) = (13, 10, -2).$$

Thus, a normal line to the tangent plane of S at P is given by $(13, 10, -2)$, and using the fact that P is on the surface, an equation of the plane is given by

$$13(x - 2) + 10(y - 1) - 2(z + 1) = 0,$$

or, equivalently,

$$13x + 10y - 2z = 38.$$

b). The plane $z = -1$ is of course the set of points with $G(x, y, z) = 0$ where $G(x, y, z) = z + 1$. We then compute the gradient:

$$\nabla G = (0, 0, 1),$$

which is the same for all points P. Thus, at P, the normal lines to S and the plane $z = -1$ are $(13, 10, -2)$ and $(0, 0, 1)$, respectively. Thus, the tangent line to the curve of intersection between S and this plane is parallel to the cross product $(13, 10, -2) \times (0, 0, 1) = (10, -13, 0)$. We can thus take $(10, -13, 0)$ as a tangent vector, and a point on the curve of intersection is of course $P = (2, 1, -1)$, so that parametric equations for the tangent line are given by

$$\begin{cases} x = 2 + 10t \\ y = 1 - 13t \\ z = -1. \end{cases}$$

Alternatively, we could also have explicitly written an equation for the curve of intersection. That is, using $z = -1$ gives $x^3y + xy = 10$, or $y(x^3 + x) = 10$.

Implicitly differentiating gives

$$y'(x^3 + x) + y(3x^2 + 1) = 0,$$

which gives $y' = -13/10$ when $x = 2$, $y = 1$ is plugged in. Using that fact that the curve, and hence its tangent line, lie in the plane $z = -1$, along with this slope of the line and the given point on the line immediately allows one to recover the above system of parametric equations for it.

(2) Suppose $f(x, y)$ is a function with critical points at $(1, -1)$, $(-1, 1)$, and $(0, 0)$ and with second order partial derivatives given by

$$f_{xx} = 12x^2 - 2y^2, \qquad f_{yy} = -2x^2 + 12y^2, \qquad f_{xy} = f_{yx} = -4xy + 2.$$

For all three critical points, determine whether the second derivative test says that the function has a local max, local min, saddle point or gives no information.

Solution: We have to look at the sign of the number $D = f_{xx}f_{yy} - f_{xy}^2$ at all three points. For the first two points $(1, -1)$ and $(-1, 1)$, $f_{xx} = 12 - 2 = 10$, $f_{yy} = -2 + 12 = 10$, and $f_{xy} = 4 + 2 = 6$, so $D = 100 - 36 = 64 > 0$, and so f has local extrema at these two points. To find out whether they are maxes or mins, we have to look at the sign of f_{xx}. In both cases, as we saw, $f_{xx} = 10 > 0$, and so there is a local minimum at both points.

At the third critical point $(0, 0)$, $f_{xx} = 0$, $f_{yy} = 0$, and $f_{xy} = 2$, and so $D = -4 < 0$. Hence, f has a saddle point at $(0, 0)$.

(3) The Extreme Value Theorem guarantees that the continuous function $f(x, y) = x\sin(y)$ has a global maximum value and a global minimum value in the square region (box) $\{(x, y) : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$. Find these global maximum and minimum values.

Solution: We first find the critical points of f. Since f is differentiable everywhere, these will only be at those points where $f_x = f_y = 0$. Taking these derivatives, this means that we need to solve the system of equations

$$\begin{cases} f_x = \sin y = 0 \\ f_y = x\cos y = 0. \end{cases}$$

Now there are two cases. If $x = 0$, then $f_y = 0$, and $f_x = 0$ if $\sin(y) = 0$. This happens if y is an integer multiple of π, but inside our box $-1 \leq y \leq 1$, and so the only critical point inside the box is $(0, 0)$. If $x \neq 0$, then all critical points have $\sin y = \cos y = 0$, which never happens (in particular, $\sin^2 y + \cos^2 y = 1$ for all y). Thus, $(0, 0)$ is the only critical point we have to consider. The value of the function at this point is $f(0, 0) = 0$.

We now look at the function on the boundary, which consists of four pieces. These are the four sides of the square surrounding the box, and are the sets

$$B_1 = \{(x, y) : x = 1,\ -1 \leq y \leq 1\},$$

$$B_2 = \{(x, y) : x = -1,\ -1 \leq y \leq 1\},$$

$$B_3 = \{(x, y) : -1 \le x \le 1,\ y = 1\},$$
$$B_4 = \{(x, y) : -1 \le x \le 1,\ y = -1\},$$

On B_1, we have to maximize

$$g(y) = f(1, y) = \sin y$$

on $[-1, 1]$. The critical points of g occur when $g'(y) = \cos y = 0$, which doesn't happen on the interval $[-1, 1]$. Thus, the extrema on this boundary piece occur at the endpoints. Hence, we compute $g(-1) = \sin(-1) = -\sin(1)$ and $g(1) = \sin(1)$, and see that on B_1, the minimum value of f is $-\sin(1)$ and the maximum value is $\sin(1)$. Similarly, on B_2, we have to optimize the function $g(y) = f(-1, y) = -\sin(y)$ on $[-1, 1]$. In this case, $g'(y) = -\cos y$, which never vanishes on $[-1, 1]$, and so the maximum and minimum on B_2 also occur at the endpoints and give a maximum value of $g(-1) = -\sin(-1) = \sin(1)$ and a minimum value of $g(1) = -\sin(1)$.

On B_3, we have to optimize the function

$$g(x) = f(x, 1) = x \sin(1)$$

on $[-1, 1]$. This is just a linear function, and so it is clear (or can of course be seen by checking that $g'(x) = \sin(1)$ never vanishes and by comparing the values at the endpoints) that the minimum value of f on B_3 is $-\sin(1)$ and the maximum value is $\sin(1)$. Similarly, on B_4, we have to optimize the function

$$g(x) = f(x, -1) = -x \sin(1),$$

which has the same maximum and minimum values.

Thus, by comparing the values of f at all critical points (in this case, only at $(0, 0)$) and all global extrema on the pieces of the boundary, we see that the maximum value of f in our square region is $\sin(1)$, and that the minimum value is $-\sin(1)$.

Advanced Problem: As mentioned in class, for functions $f(x_1, \ldots, x_n)$ of more than two variables, critical points occur in general where the gradient $\nabla f = (f_{x_1}, \ldots, f_{x_n})$ is the zero vector (or is undefined). For such points, there is an analogous second derivative test to determine whether these critical points are local minima, local maxima, saddle points, or neither. This involves the *Hessian matrix*, the matrix of all second order partial derivatives

$$\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}.$$

That is, the (i, j)-th entry of this matrix is $f_{x_ix_j}$. Generalizing the one and two-variable cases we already know, the general second derivative test states that f has a local maximum if all eigenvalues of this matrix are negative, that f

has a local minimum if all eigenvalues are positive, has a saddle point if some eigenvalues are negative and some are positive, but none are vanishing, and if at least one eigenvalue vanishes (i.e., the matrix has determinant zero), then the test is inconclusive.

Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Of course, geometrically, f is the distance of the point (x, y, z) from the origin, so clearly f has a minimum at the origin. However, forget that you know this fact, and use the above test to find all critical points of f and classify them using the second derivative test.

Solution: The gradient in this case is $\nabla f = (2x, 2y, 2z)$. This vanishes when $2x = 2y = 2z = 0$, or just when $x = y = 0$. Hence, the only critical point is at the origin. Now we compute the Hessian matrix. The first order partial derivatives are

$$f_x = 2x, \qquad f_y = 2y, \qquad f_z = 2z,$$

as we have seen, and so the Hessian matrix is (note that all mixed partial derivatives are zero)

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

As this matrix is diagonal, we can read off its eigenvalues as $2, 2, 2$. Thus, all eigenvalues (just the eigenvalue 2 with multiplicity 3) are positive, and so the function has a local minimum at $(0, 0, 0)$, as claimed.