TUTORIAL 6

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Consider the surface S given by the equation $x^3y - xyz = 10$.
   (a) Find the equation for the tangent plane to this surface at $P = (2, 1, -1)$.
   (b) Find parametric equations for the tangent line of the curve of intersection of the surface S with the plane $z = -1$ at the same point $P = (2, 1, -1)$.

(2) Suppose $f(x, y)$ is a function with critical points at $(1, -1)$, $(-1, 1)$, and $(0, 0)$ and with second order partial derivatives given by

$$f_{xx} = 12x^2 - 2y^2, \qquad f_{yy} = -2x^2 + 12y^2, \qquad f_{xy} = f_{yx} = -4xy + 2.$$

For all three critical points, determine whether the second derivative test says that the function has a local max, local min, saddle point or gives no information.

(3) The Extreme Value Theorem guarantees that the continuous function $f(x, y) = x\sin(y)$ has a global maximum value and a global minimum value in the square region (box) $\{(x, y) : -1 \le x \le 1,\ -1 \le y \le 1\}$. Find these global maximum and minimum values.

Advanced Problem: As mentioned in class, for functions $f(x_1, \dots, x_n)$ of more than two variables, critical points occur in general where the gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$ is the zero vector (or is undefined). For such points, there is an analogous second derivative test to determine whether these critical points are local minima, local maxima, saddle points, or neither. This involves the *Hessian matrix*, the matrix of all second order partial derivatives

$$\begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}.$$

That is, the (i, j)-th entry of this matrix is $f_{x_ix_j}$. Generalizing the one and two-variable cases we already know, the general second derivative test states that f has a local maximum if all eigenvalues of this matrix are negative, that f has a local minimum if all eigenvalues are positive, has a saddle point if some eigenvalues are negative and some are positive, but none are vanishing, and if at least one eigenvalue vanishes (i.e., the matrix has determinant zero), then the test is inconclusive.

Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Of course, geometrically, f is the distance of the point (x, y, z) from the origin, so clearly f has a minimum at the origin. However, forget that you know this fact, and use the above test to find all critical points of f and classify them using the second derivative test.