TUTORIAL 2

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Consider the parametric curve

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = \frac{2}{3}t^{\frac{3}{2}}. \end{cases}$$

(a) Using the base point $t_0 = 0$, find, as a function of t, the arc length s of the curve from 0 to t.
(b) Write the arc length parameterization of the curve above.
(c) Find the coordinates of the point on the curve which is an arc length distance of $\frac{14}{3}$ away from the point at $t = 0$ (in the direction of the orientation induced by the parameterization above).

(2) In this problem, you will prove the important *Frenet-Serret formulas*, which describe how the TNB frame of unit tangent, unit normal, and binormal vectors of a curve in $\mathbb{R}^3$ change as you move along the curve. These state the following (using arc length parameterization):

$$\frac{dT}{ds} = \kappa N,$$

$$\frac{dN}{ds} = -\kappa T + \tau B,$$

$$\frac{dB}{ds} = -\tau N.$$

Here, τ is the *torsion* (which we briefly saw on the homework), and κ is the curvature of the curve. More succinctly, we can use matrix multiplication notation to write (where $'$ denotes differentiation with respect to s)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

(a) Show the first equation, namely

$$\frac{dT}{ds} = \kappa N,$$

using the definition

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|},$$

and the following two facts from class:

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right|,$$

$$\kappa = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{dr}{dt}\right|}.$$

(b) Show that $\frac{dB}{ds}$ is perpendicular to B. Now show that $\frac{dB}{ds}$ is also perpendicular to T (hint: recall that B is perpendicular to both T and N by its definition as a cross product $B = T \times N$ and differentiate the equation $0 = B \cdot T$). Conclude that

$$\frac{dB}{ds} = -\tau N,$$

where τ is a scalar-valued function (the minus sign is unimportant, and only there for historical reasons).

(c) Show, by differentiating the equation $N = B \times T$ that for the same function τ you defined by the equation in b), we have

$$\frac{dN}{ds} = -\kappa T + \tau B.$$

(3) Find the curvature of the plane curve parameterized by $r(t) = (t, t^2)$ at the point when $t = 2$.