HOMEWORK 7 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Using the method of Lagrange multipliers, find the largest and smallest values of the function $f(x,y) = xy$ on the ellipse $\frac{x^2}{2} + \frac{y^2}{3} = 1$.

Solution: The ellipse is a compact set, and $f(x,y)$ is a continuous function, so we know that there will be a maximum and minimum value of f on the ellipse. Thus must occur at critical points, as there are no boundary points of the ellipse. We use the method of Lagrange multipliers to find these extrema. We want to extremize the function $f(x,y)$ subject to the constraint $g(x,y) = x^2/2 + y^2/3 - 1 = 0$. Note that

$$\nabla g = \left(x, \frac{2y}{3}\right),$$

which is only zero if $x = y = 0$. In particular, $\nabla g \neq 0$ on the ellipse. Thus, we need to solve the system of equations $\{\nabla f = (y,x) = \lambda \nabla g, \quad g = 0\}$. Comparing components of the gradients, this is equivalent to the system

$$\begin{cases} y = \lambda x \\ x = \frac{2}{3}\lambda y \\ \frac{x^2}{2} + \frac{y^2}{3} = 1. \end{cases}$$

Plugging the first equation into the second gives

$$x = \frac{2}{3}\lambda^2 x,$$

which implies $x(2\lambda^2/3 - 1) = 0$. If $x = 0$, then the first equation in the system above implies that $y = 0$, which gives the point $(x,y) = (0,0)$, which isn't on the ellipse. Thus, we must have $2\lambda^2/3 - 1 = 0$, or $\lambda = \pm\sqrt{3/2}$. Thus, the first equation above gives

$$y = \pm\sqrt{\frac{3}{2}}x.$$

Plugging into the equation for the ellipse gives

$$\frac{x^2}{2} + \frac{3}{2 \cdot 3}x^2 = x^2 = 1.$$

Thus, $x = \pm 1$. Using the equation $y = \pm\sqrt{\frac{3}{2}}x$ again gives $y = \pm\sqrt{\frac{3}{2}}$. Thus, there are 4 points which solve the system above, $(1, \sqrt{\frac{3}{2}})$, $(-1, -\sqrt{\frac{3}{2}})$, $(1, -\sqrt{\frac{3}{2}})$, and $(-1, \sqrt{\frac{3}{2}})$. In the first two cases, $xy = \sqrt{\frac{3}{2}}$, which is then the maximum value of f, while f attains its minimum value at the second two points, namely $-\sqrt{\frac{3}{2}}$.

Alternatively, and more messily, we could write the equation of the ellipse as

$$y = \pm\sqrt{-\frac{3x^2}{2} + 3},$$

which when plugged into $f(x, y)$ gives

$$g(x) = f\left(x, \pm\sqrt{-\frac{3x^2}{2} + 3}\right) = \pm x\sqrt{-\frac{3x^2}{2} + 3}.$$

This would have to be extremized for x in $[-\sqrt{2}, \sqrt{2}]$ (using the fact that (x, y) is on the ellipse again). On the boundary points, $x = \pm\sqrt{2}$, we have $y = 0$, and so $f(x, y) = 0$. One can differentiate g to get (after a little rearrangement)

$$g'(x) = \pm\frac{3(x^2 - 1)}{\sqrt{-\frac{3x^2}{2} + 3}},$$

which clearly vanishes if and only if $x = \pm 1$. This gives the same four points as we found using Lagrange multipliers above. However, the idea was to demonstrate the difference between the two methods and how Lagrange multipliers makes things look much simpler and more symmetric in many cases. For instance, in this case, although the ellipse isn't the graph of a function of x, it nearly is. Specifically, it is the union of two graphs. However, solving for y, even using multiple functions, for more general curves may not even be possible, so this direct method of solving for y in the curve and plugging into the function to be extremized is often infeasible or impossible.

(2) Find the absolute maximum and minimum values of $f(x, y) = x^2 - y^2$ on the closed unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$.

Solution:

The Extreme Value Theorem guarantees that these extrema exist. We have to look for critical points on the interior (where $x^2 + y^2 < 1$), and for potential extrema on the boundary (the circle $x^2 + y^2 = 1$). On the interior, we want to solve $\nabla f = (2x, -2y) = (0, 0$. Clearly, this only happens at $(x, y) = (0, 0)$. At this point, $f(0, 0) = 0$. This is the only potential extremum on the interior. On the boundary circle, we can use the method of Lagrange multipliers, as in problem (1), to extremize the function $f(x, y)$ subject to the constraint $g(x, y) =$

$x^2 + y^2 - 1 = 0$. For this, note that $\nabla g = (2x, 2y) \neq (0, 0)$ on the circle (as the origin ins't on our circle). Thus, we have to solve $\nabla f = \lambda g$ and $g = 0$. This gives the system of equations

$$\begin{cases} x = \lambda x \\ y = -\lambda y \\ x^2 + y^2 = 1. \end{cases}$$

There are several cases. If $x = 0$, then since $x^2 + y^2 = 1$, we have $y = \pm 1$. Thus, two possible extrema are at $(0, -1)$ and $(0, 1)$. The value of f at both these points is $f(0, \pm 1) = -1$. Now, if $x \neq 0$, then the equation $x = \lambda x$ implies that $1 = \lambda$. But then the equation $y = -\lambda y$ becomes $y = -y$, and so $y = 0$, $x = \pm 1$. The value of the function f at these two points is $f(\pm 1, 0) = 1$. Comparing all these values with the value of the function at the critical point in the origin identifies in the last step, we see that the global min value of f on the closed unit disc is -1 and that the maximum value is $+1$.

(3) As hinted at in class, there is a method of Lagrange multipliers not only for functions of many variables but for functions with multiple constraints. We will state the system of equations for a function subject to 2 constraints just for concreteness, but you should be able to guess the general formula from this example and the cases you have already seen! In this case, the system of equations to solve to search for extrema for a function $f(x_1, \ldots, x_n)$ subject to the constraints $g(x_1, \ldots, x_n) = 0$ and $h(x_1, \ldots, x_n) = 0$ is

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 0. \end{cases}$$

Here, λ and μ are two real numbers. Note that there are $n + 2$ unknowns ($x_1, \ldots, x_n$ and λ, μ), and $n + 2$ equations in this system.

Use this method to find the closest point on the intersection of the surfaces $x^2 + y^2 = z^2$ and $x + y - z = 2$ to the origin.

Solution: We need to extremize the distance function between an arbitrary point (x, y, z) and the origin. To simplify, we instead look for the minimum of the square of the distance function, namely

$$f(x, y, z) = x^2 + y^2 + z^2.$$

So we want to use the method of Lagrange multipliers to minimize f subject to the constraints $g = 0$ and $h = 0$, where $g(x, y, z) = x^2 + y^2 - z^2$, $h(x, y, z) = x + y - z - 2$. We then have

$$\nabla f = (2x, 2y, 2z), \qquad \nabla g = (2x, 2y, -2z), \qquad \nabla h = (1, 1, -1).$$

Thus, the method of Lagrange multipliers gives the system of equations (note that we can factor the 2 out of ∇f and ∇g for convenience, which gets absorbed into the arbitrary constant μ)

$$\begin{cases} x = \lambda x + \mu \\ y = \lambda y + \mu \\ z = -\lambda z - \mu \\ x^2 + y^2 = z^2 \\ x + y - z = 2. \end{cases}$$

First suppose that $\lambda = 1$. If it is, then $x = 1 \cdot x + \mu$, so that $\mu = 0$. But then the third equation in our system implies that $z = -z$. Then $z = 0$, and the equation $x^2 + y^2 = z^2$ implies that $x^2 + y^2 = 0$, which implies that $x = y = z = 0$. But that point isn't on the plane $x + y - z = 2$.

Thus, we can assume $\lambda \neq 1$. Thus, we can solve the first equation in our system for x to find

$$x = \frac{\mu}{1-\lambda}.$$

Similarly, the second equation implies that

$$y = \frac{\mu}{1-\lambda}.$$

Thus, we have $x = y$. This drastically simplifies the remaining calculations. Since $x = y$, plugging into the equation for the plane gives $2x - z = 2$, or $z = 2x - 2$. Plugging $(x, x, 2x - 2)$ in for (x, y, z) into the equation $x^2 + y^2 = z^2$ gives

$$2x^2 = (2x - 2)^2 = 4x^2 - 8x + 4,$$

or $2x^2 - 8x + 4 = 0$, which implies $x^2 - 4x + 2 = 0$. This quadratic has roots $x = 2 \pm \sqrt{2}$. Thus, plugging back into the equations above, we find $x = y = 2 \pm \sqrt{2}$, $z = 2 \pm 2\sqrt{2}$. Plugging these values into f gives the minimum value $24 - 16\sqrt{2}$ at $x = 2 - \sqrt{2}$ (the value at $x = 2 + \sqrt{2}$ is $24 + 16\sqrt{2}$ which is larger). In fact, geometrically, the intersection of this (double) cone with this plane is a hyperbola, and both points gave local minimums which are at the closest points on each branch of the hyperbola to the origin. Thus, the minimum distance from the origin to a point on this hyperbola is $\sqrt{24 - 16\sqrt{2}}$. This geometric picture makes it clear that one of the points we found using the method of Lagrange multipliers has to be a global minimum, as there is clearly some closest point to the origin and this must occur at a critical point.

To compare with a method without using Lagrange multipliers, we could do the following. Let's parameterize this hyperbola. Set $x = t$, and using the two equations for the cone and the plane, after a little algebra we get the parametric coordinates

$$x = t, \qquad y = 2 \cdot \frac{t-1}{t-2}, \qquad z = \frac{t^2 - 2t + 2}{t-2}$$

$(t \neq 2)$. Plugging this into the (square of the) distance function gives

$$f(x(t), y(t), z(t)) = x^2 + y^2 + z^2 = t^2 + \left(2 \cdot \frac{t-1}{t-2}\right)^2 + \left(\frac{t^2 - 2t + 2}{t-2}\right)^2,$$

which after combining and simplifying is equal to (call it $g(t)$ for convenience)

$$g(t) = 2 \cdot \frac{(t^2 - 2t + 2)^2}{(t-2)^2}.$$

Differentiating and simplifying gives

$$g'(t) = 4 \cdot \frac{(t^2 - 2t + 2)(t^2 - 4t + 2)}{(t-2)^3}.$$

Critical points thus occur at $x = t = 2 + \sqrt{2}$, and $x = t = 2 - \sqrt{2}$ (note that $t = 2$ wasn't in our original domain and one of the quadratics has no real roots). Plugging into the equations for y and z recovers the points discovered by Lagrange multipliers above. Thus, in this case, it is also possible to avoid using Lagrange multipliers, but significantly more algebra has to be carried out!

(4) Evaluate the double integral

$$\int_2^3 \int_0^1 (4xy - x^2 + y)dydx.$$

Also compute

$$\int_0^1 \int_2^3 (4xy - x^2 + y)dxdy$$

directly to see that you get the same answer.

Solution: We find that

$$\begin{aligned}
\int_2^3 \int_0^1 (4xy - x^2 + y)dydx &= \int_2^3 \left(\int_0^1 (4xy - x^2 + y)dy\right) dx \\
&= \int_2^3 \left[2xy^2 - x^2y + y^2/2\right]_0^1 dx = \int_2^3 (2x - x^2 + 1/2)dx \\
&= \left[x^2 - \frac{x^3}{3} + \frac{x}{2}\right]_2^3 = \frac{3}{2} - \frac{7}{3} = -\frac{5}{6}.
\end{aligned}$$

Computing the double integral in the other order gives

$$\begin{aligned}\int_0^1 \int_2^3 (4xy - x^2 + y)dxdy &= \int_0^1 \left(\int_2^3 (4xy - x^2 + y)dx \right) dy \\ &= \int_0^1 \left[2x^2y - x^3/3 + xy\right]_2^3 dy = \int_0^1 ((21y - 9) - (10y - 8/3))dy \\ &= \int_0^1 (11y - 19/3)dy \\ &= \left[\frac{11y^2}{2} - \frac{19y}{3}\right]_0^1 = \frac{11}{2} - \frac{19}{3} = -\frac{5}{6}.\end{aligned}$$

which is the same number we got above.

(5) The average value of a two variable function $f(x, y)$ on a region R of the $x - y$ plane is the double integral of f over that region $\iint_R f(x, y)dA$, divided by the area of the region. Find the average value of $f(x, y) = x^2 + y^2 - 2xy$ on the box $[0, 1] \times [4, 5]$.

Solution: The area of this rectangle is the product of its side lengths, or $1 \cdot 1 = 1$. Thus, in this case, the average of f on the rectangle is just $\iint_R f(x, y)dA$. We will compute this by integrating with respect to x first (since it will be the easier one since the endpoints are 0 and 1 which are nice to evaluate polynomials at). That is, the average is

$$\begin{aligned}&\int_4^5 \left(\int_0^1 (x^2 + y^2 - 2xy)dx \right) dy \\ &= \int_4^5 \left[\frac{x^3}{3} + xy^2 - x^2y)\right]_0^1 dy = \int_4^5 (1/3 + y^2 - y)dy \\ &= \left[\frac{y^3}{3} - \frac{y^2}{2} + \frac{y}{3}\right]_4^5 = \frac{185}{6} - \frac{44}{3} = \frac{97}{6}.\end{aligned}$$