HOMEWORK 6 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Find the equation to the tangent plane of the surface $z = f(x, y) = xe^y$ at the point where $x = 3$, $y = 0$. Also find parametric equations of the normal line to this plane.

Solution: This surface is given by the equation $F(x, y, z) = 0$, where $F(x, y, z) = f(x, y) - z = xe^y - z$. We then compute that

$$\nabla F = (F_x, F_y, F_z) = (e^y, xe^y, -1).$$

Plugging in the point $(3, 0, 3e^0) = (3, 0, 3)$ gives

$$\nabla F(3, 0, 3) = (1, 3, -1),$$

which is a normal vector to the tangent plane. Hence, the equation of the tangent plane is

$$(x - 3) + 3y - (z - 3) = 0,$$

or

$$x + 3y - z = 0.$$

The normal line is parallel to $(1, 3, -1)$ and passes through $(3, 0, 3)$, and so can be parameterized as

$$\begin{cases} x = 3 + t \\ y = 3t \\ z = 3 - t. \end{cases}$$

(2) Describe the intersection between the two surfaces $x^2 + y^2 + z^2 = 2$ and $z^2 = x^2 + y^2$. Show that at all points in the intersection, the normal vectors of the two corresponding tangent planes are perpendicular. Further find parametric equations of the tangent line to the curve of intersection passing through $P = (1, 0, -1)$ at P.

Solution: If a point (x, y, z) is on both surfaces, then by using the second equation, $x^2 + y^2 = z^2$, and plugging into the equation defining the first surface, we find

$$z^2 + z^2 = 2z^2 = 2,$$

or $z = \pm 1$. Thus, we find that the intersection is the union of two circles, which are described by the equation $x^2 + y^2 = 1$ together with the constraint that they lie in the planes $z = \pm 1$. Geometrically, this intersection was between a sphere and a (double) cone. These two surfaces are defined by $F(x, y, z) = 0$,

$G(x, y, z) = 0$ with $F(x, y, z) = x^2 + y^2 + z^2 - 2$, $G(x, y, z) = x^2 + y^2 - z^2$. We then compute

$$\nabla F = (2x, 2y, 2z) = 2(x, y, z),$$
$$\nabla G = (2x, 2y, -2z) = 2(x, y, -z).$$

At any point on the circles of intersection, these two vectors give the normal lines of the tangent planes of the two surfaces. The two vectors are orthogonal at all such intersection points, since

$$(x, y, z) \cdot (x, y, -z) = x^2 + y^2 - z^2 = 0$$

for all such points since this last equality just says that the point lies on the cone $x^2 + y^2 = z^2$.

To find the tangent line through the circle of intersection with $z = -1$ at P, we first compute (rescaling out factors of 2 for convenience)

$$\frac{1}{2}\nabla F(1, 0, -1) = (1, 0, -1),$$
$$\frac{1}{2}\nabla G(1, 0, -1) = (1, 0, 1).$$

A vector parallel to the tangent line is then given by the cross product of these two vectors, which is $(1, 0, -1) \times (1, 0, 1) = -2(0, 1, 0)$. Thus, $(0, 1, 0)$ points in the direction of the tangent line, which also passes through $(1, 0, -1)$, and so parametric equtions for this line may be given by

$$\begin{cases} x = 1 \\ y = t \\ z = -1. \end{cases}$$

This line may also be specified by the equations $x = 1, \ z = -1$. Geometrically, we were computing the tangent line to the circle $x^2 + y^2 = 1$ in the plane $z = -1$ at the "rightmost point" (looking from above with the x-axis pointing to the right as usual), and we found that in this plane the tangent line is a "vertical" tangent line $x = 1$, which we also already knew geometrically.

(3) Find the local (aka relative) extrema and saddle points of the function

$$f(x, y) = -4x^2y + 2x^2 + y^2 - 7.$$

Solution: We compute (the first derivatives are needed to find the critical points, the second derivatives are computed now for convenience since we will need them to classify the critical points anyways)

$$f_x = -8xy + 4x,$$
$$f_y = -4x^2 + 2y,$$
$$f_{xx} = -8y + 4,$$
$$f_{yy} = 2,$$

$$f_{xy} = f_{yx} = -8x.$$

The function f is clearly differentiable (as its a polynomial), and so the critical points will occur only at those points when $f_x = f_y = 0$. Thus, we have to solve the system of equations

$$\begin{cases} -8xy + 4x = 0, \\ -4x^2 + 2y = 0. \end{cases}$$

The first equation is equivalent to $x(1-2y) = 0$. Thus, there are two cases, if $x = 0$, then by plugging into the second equation we get the critical point $(0, 0)$. If $x \neq 0$, then $1 - 2y = 0$, and so $y = 1/2$. Plugging into the second equation gives

$$-4x^2 + 1 = 0,$$

which has solutions $x = \pm\frac{1}{2}$. Thus, the 3 critical points of f are $(0, 0)$, $(1/2, 1/2)$, and $(-1/2, 1/2)$. To classify these critical points, we plug them into the expression

$$D = f_{xx}f_{yy} - f_{xy}^2 = 8 - 16y - 64x^2,$$

which at $(0, 0)$ is 8, at $(1/2, 1/2)$ is -16, and at $(-1/2, 1/2)$ is also -16. Thus, at the critical points $(\pm 1/2, 1/2)$, f has saddlepoints. At the point $(0, 0)$, $D > 0$ and so there is a local extremum. To find out whether its a maximum or a minimum we have to look at the sign of $f_{x,x}(0, 0) = 4 > 0$. Since this is > 0, the second derivative test implies that f has a local minimum at $(0, 0)$.

(4) Find the point on the plane $3x + 2y + z = 1$ closest to the point $(-1, 2, 1)$ (hint: instead of minimizing the function describing the distance between a point on the plane and $(1, -2, 1)$, minimize a related function).

Solution: The distance from an arbitrary point (x, y, z) in $\mathbb{R}^3$ to $(-1, 2, 1)$ is

$$d = \sqrt{(x+1)^2 + (y-2)^2 + (z-1)^2}.$$

To find where d is smallest, it suffices to find where

$$d^2 = (x+1)^2 + (y-2)^2 + (z-1)^2$$

is smallest, since this will be the same point. This is much nicer to work with, since it gets rid of the square root. We want to minimize a function of two variables, which can be obtained by using the constraint that (x, y, z) lies on the given plane. Thus, we can plug in $-3x - 2y$ for $z - 1$ into d^2 and so the point we are looking for is where

$$f(x, y) = (x+1)^2 + (y-2)^2 + (-3x-2y)^2 = 10x^2 + 12xy + 5y^2 + 2x - 4y + 5$$

attains a global minimum. Geometrically, it is clear that there is a unique point on the plane which is closest to the point $(-1, 2, 1)$, and in some small closed disk around this point shows that this point is on the interior of a compact region and hence lies at a critical point. In other words, we will assume, hoping that this is sufficient justification for our purposes, that this point is a local minimum of $f(x,y)$. We thus want to find the critical points of $f(x,y)$. We compute

$$f_x = 20x + 12y + 2,$$
$$f_y = 10y + 12x - 4,$$
$$f_{xx} = 20, \qquad f_{yy} = 10, \qquad f_{xy} = 12.$$

Note that at all points (x,y,z), $D = f_{xx}f_{yy} - f_{xy}^2 = 200 - 144 > 0$ and that $f_{xx} > 0$, and that $f(x,y)$ is differentiable everywhere. Thus, by the second derivative test, we see that all critical points are local minima. To find these points, we want to solve $\nabla f = 0$, which gives the system of equations

$$\begin{cases} 20x + 12y = -2, \\ 12x + 10y = 4. \end{cases}$$

Solving this linear system gives $x = -17/14,\ y = 13/7$. Plugging back into the equation of the plane gives $z = 1 - 3x - 2y = 13/14$. Thus, the point on the plane which is closest to $(-1,2,0)$ is $(-17/14, 13/7, 13/14)$.

(5) Find the absolute minimum and maximum values (guaranteed to exist by the Extreme Value Theorem) of the function

$$f(x,y) = x^2y - 3xy + x^3 + 7$$

on the triangular region (including the interior and the boundary) bounded by the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,-1)$.

Solutions We first find the critical points inside of the triangle. We compute

$$f_x = 2xy - 3y + 3x^2,$$
$$f_y = x^2 - 3x,$$
$$f_{xx} = 2y + 6x, \qquad f_{yy} = 0, \qquad f_{xy} = 2x - 3.$$

As this function is differentiable everywhere, critical points will occur when $\nabla f = 0$, i.e., when

$$\begin{cases} 2xy + 3x^2 - 3y = 0, \\ x^2 - 3x = 0. \end{cases}$$

The solutions to the second equation are $x = 0, 3$, which by plugging back into the first equation are seen to give rise to the critical points $(0,0)$ and $(3,-9)$. As the first point is already on the boundary of our triangle (and hence doesn't need to be considered separately from the analysis of the boundary case we will do next anyways) and the point $(3,-9)$ doesn't lie within out triangle, the global extrema for f in the closed triangular region lie on the boundary; i.e., on the actual triangle.

The triangle consists of three pieces (the three sides). The first one is the set of points $(0,y)$ with $-1 \le y \le 0$. Call this set B_1. The second is the set of points

$(x, 0)$ with $0 \leq x \leq 2$. Call this set B_2. The final side of the triangle is the line segment from $(2, 0)$ to $(0, -1)$. The line through these two points has slope $1/2$, and can thus be written as $y = x/2 - 1$. That is, the last subset of the boundary, B_3, is the set of points $(x, x/2 - 1)$ with $0 \leq x \leq 2$.

We now find the global extrema of f restricted to these three line segments. We start with B_1. Then we want the extrema of

$$g_1(y) = f(0, y) = 7$$

on the interval $[-1, 0]$. Since this is constant, the global maximum value and global minimum value on B_1 are both given by 7.

We next move to B_2. We want to find the extrema of

$$g_2(x) = f(x, 0) = x^3 + 7$$

on the interval $[0, 2]$. The critical points of g_2 occur when $g_2'(x) = 3x^2 + 7 = 0$, which doesn't hold for any real numbers. Thus, the global maximum and minimum values of g_2 on the interval are obtained at the endpoints. We then have $g_2(0) = 7$, $g_2(2) = 15$. Hence, the minimum value of f on B_2 is 7 and the maximum value is 15.

Finally, we look at B_3. We need to find the extrema of the function

$$g_3(x) = f(x, x/2 - 1) = \frac{3x^3}{2} - \frac{5x^2}{2} + 3x + 7$$

on the interval $[0, 2]$. To find the critical points, compute $g_3'(x) = \frac{9x^2}{2} - 5x + 3$. The discriminant of this quadratic is $25 - 18 \cdot 3 < 0$, and so g_3 has no critical points. Thus, it attains its global extrema at the endpoints of the interval. We compute $g_3(0) = 7$, and $g_3(0) = 15$ (note that this isn't really necessary, as these endpoints already correspond with endpoints of the regions B_1 and B_2 which were computed above). Thus, the global minimum value of f on B_3 is 7, and the maximum on B_3 is 15.

Comparing all the values computed above, we see that the global minimum value of $f(x, y)$ on the triangular region is 7, and the global maximum value is 15.