HOMEWORK 4

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) In class, we considered the piecewise function

$$f(x,y) = \begin{cases} \frac{-xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Compute the partial derivatives f_x and f_y and show that they exit everywhere. Conclude that this function has both first-order partial derivatives existing everywhere but that the original function isn't continuous everywhere (and hence, the function isn't differentiable).

(2) You are standing on a hill shaped like the torus defined by the equation

$$(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2), \quad z \geq 0,$$

where z points in the upwards direction, x in the eastwards direction, and y in the northwards direction. If you are at the point $(3/2, 0, \sqrt{3}/2)$, then what is the angle of ascent that your path takes if you travel straight east?

(3) The one-dimensional *wave equation*, which describes the propagation of waves with respect to space and time, is for a function $u(x,t)$, which is meant to describe the shape of the wave at each point in time t, we have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Here, $c > 0$ is a constant depending on the wave. This is an example of a *partial differential equation* (PDE), which often arise naturally in physics. Although such equations are typically very hard to solve, in this case, it is reasonable to do so, and in fact the general solution is given by

$$u(x,t) = F(x - ct) + G(x + ct)$$

for one-variable functions F, G. Show that for any twice-differentiable functions F and G, that this is indeed a solution of the wave equation. Moreover, show that if a solution to the equation satisfies the initial conditions

$$U(x,0) = f(x),$$
$$U_t(x,0) = g(x),$$

then the function

$$U(x,t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

solves the wave equation with the given initial conditions.

(4) Using a two-variable linear approximation around the point $(x_0, y_0) = (4, 2)$, approximate the value $f(3.9, 2.05)$ where

$$f(x, y) = 3x^2y - \frac{x}{y}.$$

(5) Find the equation of the tangent plane to the surface

$$x^2 + y^2 - 4 = z$$

at $(2, -2, 0)$.