HOMEWORK 3

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) A particle moves along a curve in $\mathbb{R}^3$ with position function given by $r(t) = (e^t, \sqrt{t^2+1}, t)$. Find the velocity $v(t)$, the acceleration $a(t)$, the speed as a function of time, and the curvature κ. Further find the tangent and normal components of acceleration, a_T and a_N respectively, as functions of time.

Solution: We differentiate $r(t)$ to find

$$v(t) = r'(t) = \left(e^t, \frac{t}{\sqrt{t^2+1}}, 1\right),$$

and differentiate again to obtain

$$a(t) = v'(t) = r''(t) = \left(e^t, \frac{1}{(t^2+1)^{\frac{3}{2}}}, 0\right).$$

The speed is given by

$$|v(t)| = \sqrt{e^{2t} + \frac{t^2}{t^2+1} + 1}.$$

Thus,

$$a_T = \frac{v \cdot a}{|v|} = \frac{e^{2t} + \frac{t}{(t^2+1)^2}}{\sqrt{e^{2t} + \frac{t^2}{t^2+1} + 1}}.$$

To find a_N, we first compute

$$v \times a = \det \begin{pmatrix} i & j & k \\ e^t & \frac{t}{\sqrt{t^2+1}} & 1 \\ e^t & \frac{1}{(t^2+1)^{\frac{3}{2}}} & 0 \end{pmatrix} = \left(-\frac{1}{(t^2+1)^{\frac{3}{2}}}, e^t, -\frac{e^t}{(t^2+1)^{\frac{3}{2}}}\left(t^3+t-1\right)\right),$$

and so

$$|v \times a| = \sqrt{\frac{1 + e^{2t}(t^3+t-1)^2}{(t^2+1)^3} + e^{2t}}.$$

Therefore,

$$a_N = \frac{|v \times a|}{|v|} = \sqrt{\frac{\frac{1+e^{2t}(t^3+t-1)^2}{(t^2+1)^3} + e^{2t}}{e^{2t} + \frac{t^2}{t^2+1} + 1}}$$

and

$$\kappa = \frac{|v \times a|}{|v|^3} = \sqrt{\frac{\frac{1+e^{2t}(t^3+t-1)^2}{(t^2+1)^3} + e^{2t}}{\left(e^{2t} + \frac{t^2}{t^2+1} + 1\right)^3}}.$$

(2) Suppose a particle travels through $\mathbb{R}^3$ with position vector given by $r(t) = \left(t\sqrt{2}, \frac{t^2}{2}, \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\right)$. Find the distance the particle travels from $t = 0$ to $t = \sqrt{3} - 1$ (hint: use a trig integral and then integration by parts, and recall that $\int \sec\vartheta d\vartheta = \log|\sec\vartheta + \tan\vartheta| + C$).

Solution: The velocity of the particle is

$$v(t) = r'(t) = (\sqrt{2}, t, \sqrt{2t}),$$

and so the speed is

$$|v(t)| = \sqrt{2 + t^2 + 2t}.$$

The distance travelled is thus

$$\int_0^{\sqrt{3}-1} |v(t)|dt = \int_0^{\sqrt{3}-1} \sqrt{t^2 + 2t + 2}\ dt.$$

To compute this integral, we want to use a trig substitution, but first we must complete the square, and write our integral as

$$\int_0^{\sqrt{3}-1} \sqrt{(t+1)^2 + 1}\ dt.$$

This suggests that we substitute $t = \tan\vartheta - 1$ (with $-\pi/2 < \vartheta < \pi/2$), so that $dt = \sec^2\vartheta d\vartheta$ and so

$$\sqrt{(t+1)^2 + 1}\ dt = \sqrt{\tan^2\vartheta + 1}\sec^2\vartheta d\vartheta = \sec^3\vartheta d\vartheta$$

(in the last equality, note that we used the fact that $\sec\vartheta > 0$ in the specified range). Thus, our integral becomes

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3\vartheta d\vartheta.$$

This integral can be found by integrating by parts with

$$u = \sec\vartheta, \quad du = \sec\vartheta\tan\vartheta d\vartheta, \quad dv = \sec^2\vartheta d\vartheta, \quad v = \tan\vartheta.$$

Hence,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3 \vartheta d\vartheta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} u dv = [uv]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} v du$$

$$= [\sec\vartheta \tan\vartheta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\vartheta \tan^2 \vartheta d\vartheta = [\sec\vartheta \tan\vartheta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\vartheta (\sec^2\vartheta - 1) d\vartheta$$

$$= [\sec\vartheta \tan\vartheta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\vartheta d\vartheta - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3 \vartheta d\vartheta$$

$$= [\sec\vartheta \tan\vartheta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + [\log|\sec\vartheta + \tan\vartheta|]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3 \vartheta d\vartheta$$

$$= (2\sqrt{3} - \sqrt{2}) + (\log(\sqrt{3}+2) - \log(\sqrt{2}+1)) - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3 \vartheta d\vartheta.$$

Solving for the integral we want (comparing the very first and last expressions in the preceding chain of equalities), we find

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^3 \vartheta d\vartheta = (\sqrt{3} - 2^{-\frac{1}{2}}) + \frac{1}{2}(\log(\sqrt{3}+2) - \log(\sqrt{2}+1)),$$

which is the distance the particle travelled.

(3) You are standing on the edge of a Hag's Head, which is a cliff 120 m high above the Atlantic. Out of frustration with the lack of cell service, you throw your phone towards the ocean at an angle of 45° at a speed of 10 m/s. Find parametric equations describing the trajectory of the phone, and determine how far it travels horizontally before it hits the water.

Solution: In general, the parametric equations of such projectile motion are given by

$$\begin{cases} x(t) = v_0 \cos\alpha t, \\ y(t) = s_0 + v_0 \sin\alpha t - \frac{1}{2} g t^2, \end{cases}$$

where v_0 is the initial speed, s_0 is the initial height, α is the angle of the initial velocity makes with respect to the horizontal, and g is about 9.8 m/s^2 on Earth, which we will just approximate to be 10 m/s^2 in this example. In our case, this becomes

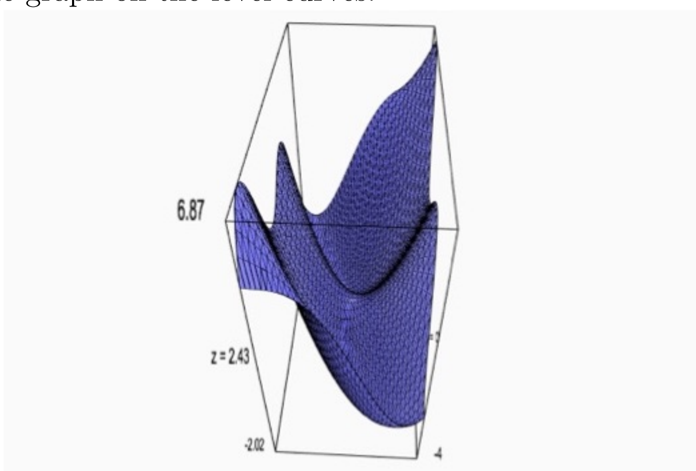
$$\begin{cases} x(t) = 5\sqrt{2}t, \\ y(t) = 120 + 5\sqrt{2}t - 5t^2. \end{cases}$$

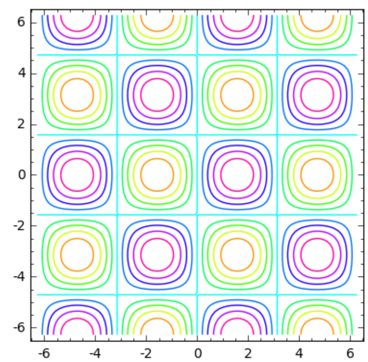
The phone hits the water when $y(t) = 0$, which gives

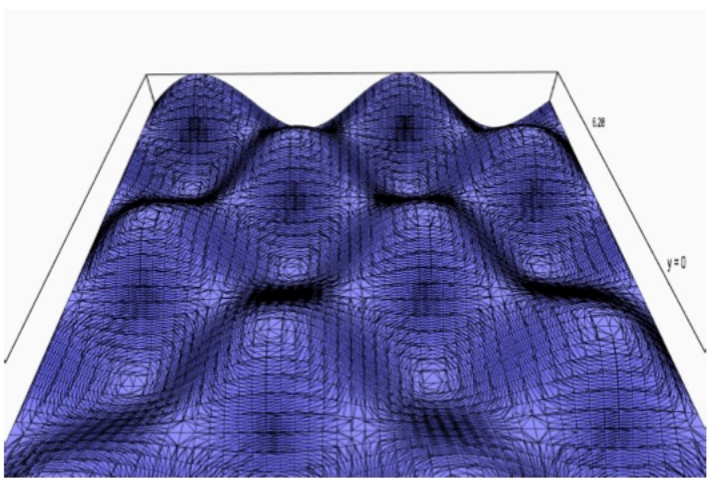
$$t = \frac{5\sqrt{2} \pm \sqrt{2450}}{10}.$$

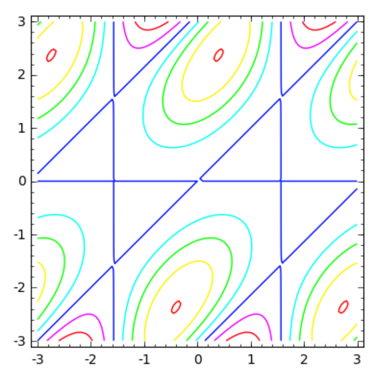
Obviously, only the positive solution $t = 4\sqrt{2}$ is relevant. Plugging in gives $x(4\sqrt{2}) = 40$, so the phone has been thrown approximately 40 m into the ocean (without rounding g, the more accurate answer is about 40.4 m, which is pretty close to this estimate).

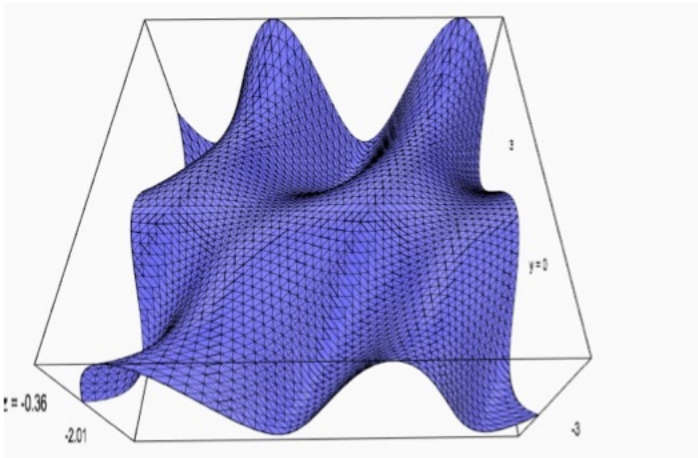
(4) Match the following plots of two-variable functions to their contour plots. The colors of the lines in the contour plots give an indication of the relative height of the graph on the level curves.

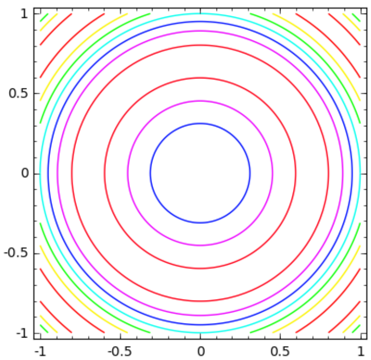


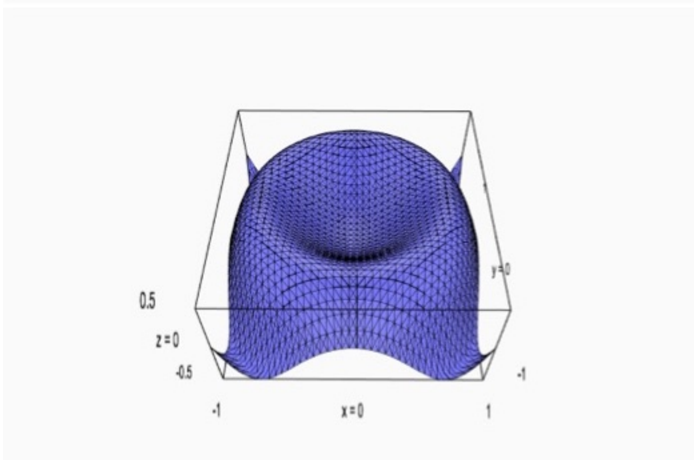


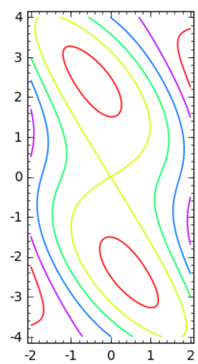




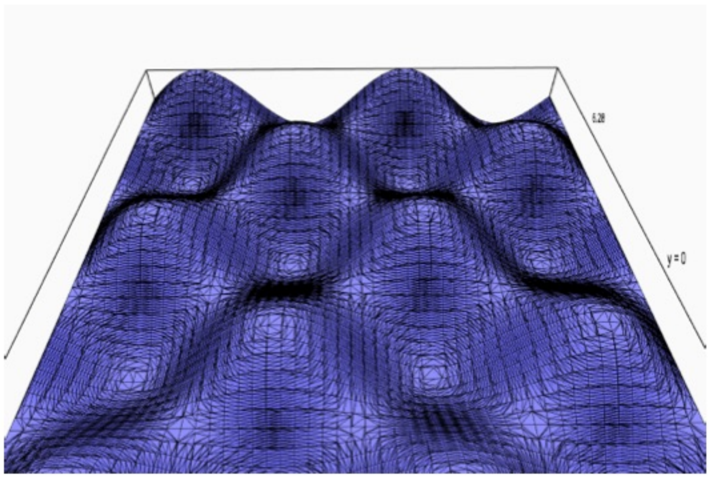


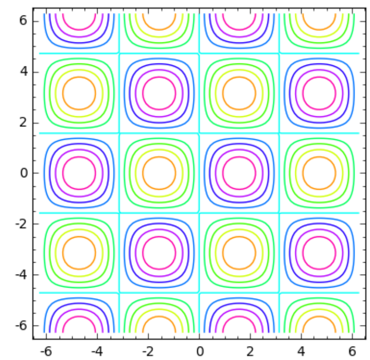


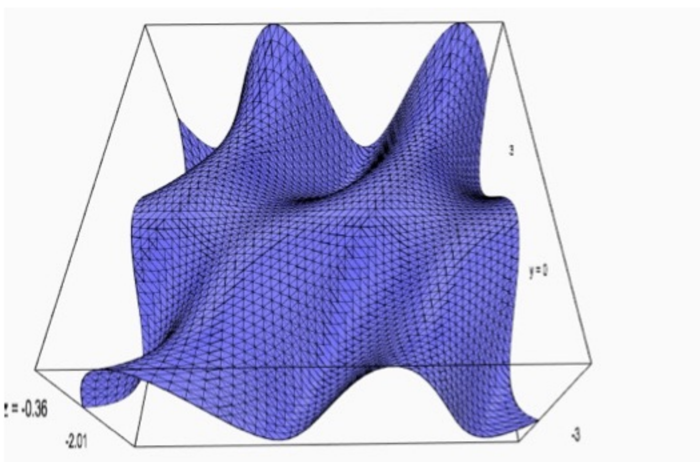


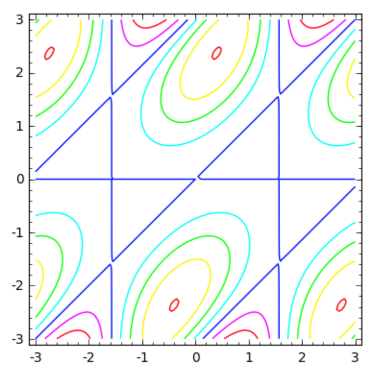


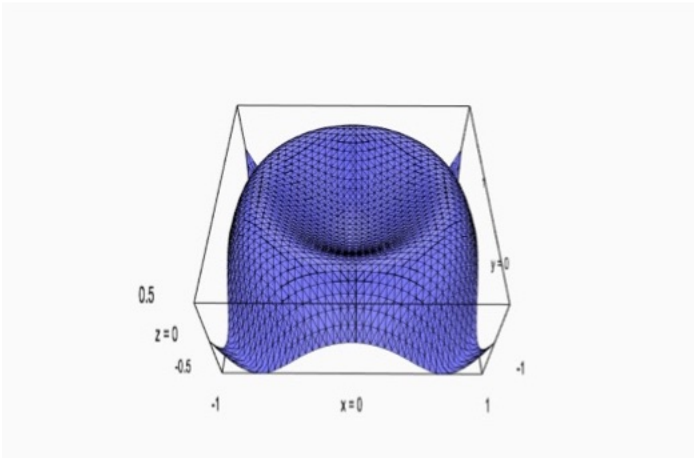
Solution: The correct matching is as follows:

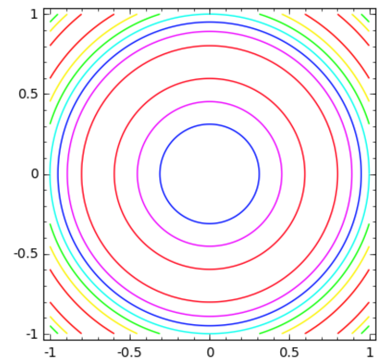

y = 0

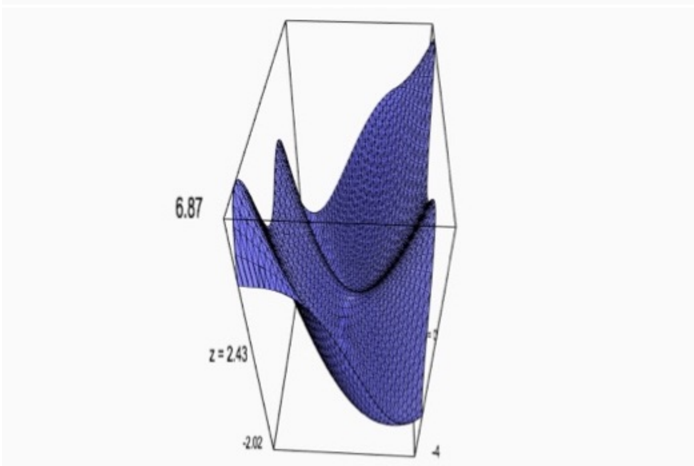

6
4
2
0
-2
-4
-6
-6 -4 -2 0 2 4 6

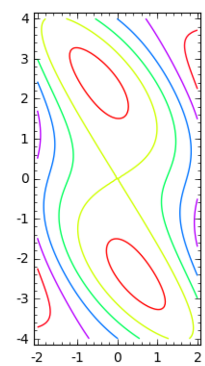

y = 0
z = -0.36
-2.01
-3


3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3





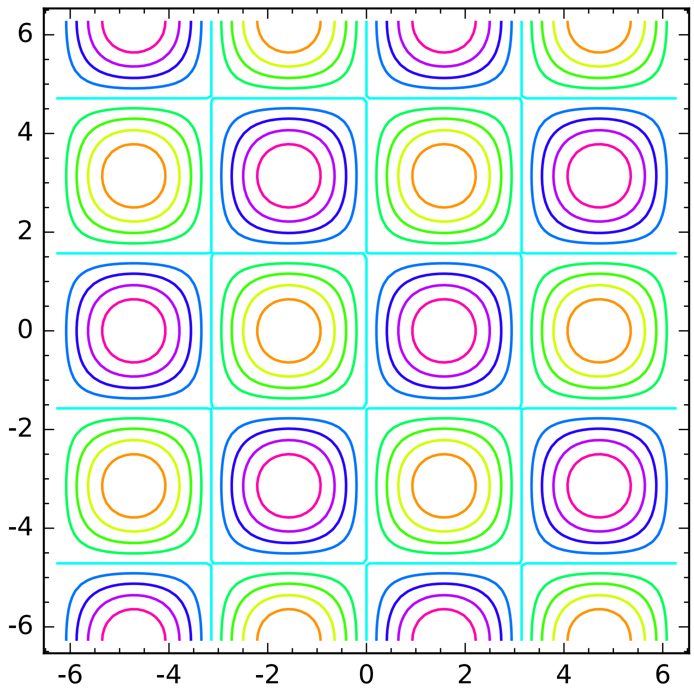




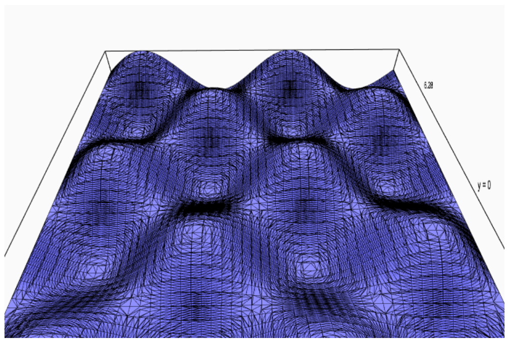
For those who are interested in playing around in SAGE to see these examples working and to be able to rotate the 3D figures interactively, the code used to generate the images is given below.

```
var('x,y,z')
contour_plot(sin(x)*cos(y), (x,-2*pi,2*pi), (y,-2*pi,2*pi),contours\
    =[-1,-4/5,-3/5,-2/5,-1/5,0,1/5,2/5,3/5,4/5,1],fill=false,cmap='\
    hsv')
```

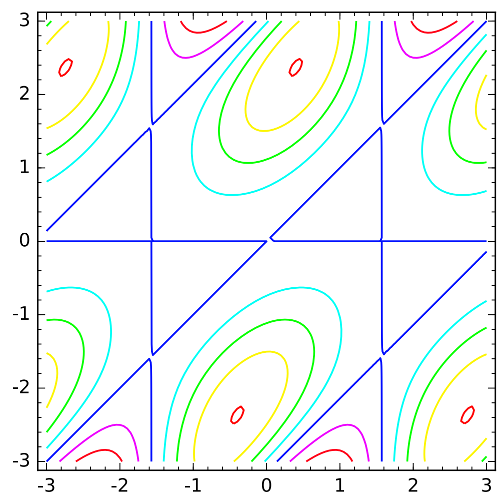
(x, y, z)



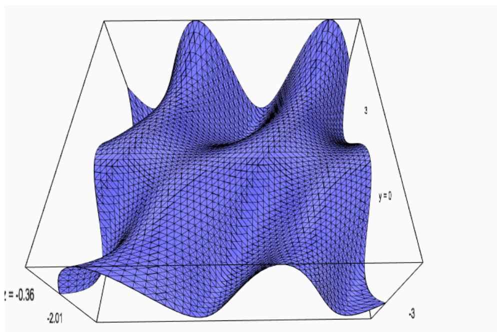

```
implicit_plot3d(sin(x)*cos(y)==z,(x,-2*pi,2*pi),(y,-2*pi,2*pi),(z\
    ,-1,1),mesh=true,plot_points=50)
```



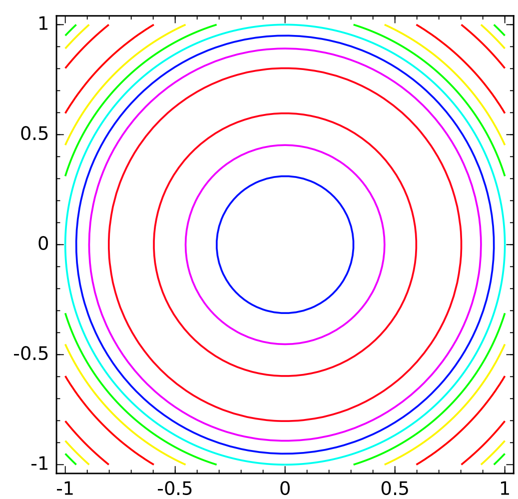

```
contour_plot(sin(x-y)*y*cos(x), (x,-3,3), (y,-3,3),plot_points=121,\
    fill=false,cmap='hsv')
```



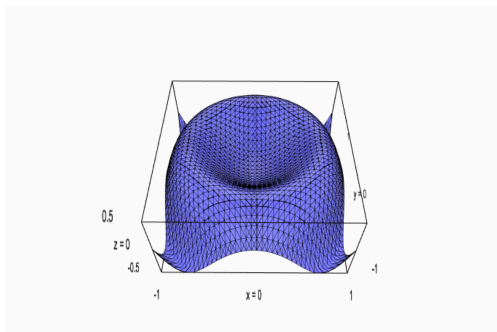
```
plot3d(sin(x-y)*y*cos(x),(x,-3,3), (y,-3,3) ,mesh=true,plot_points\
    =50)
```



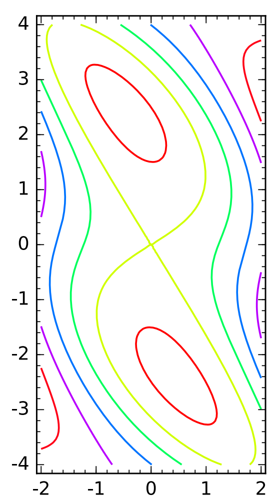

```
contour_plot(sin(pi*(x^2+y^2))/2,(x,-1,1),(y,-1,1),plot_points=121,\
    fill=false,cmap='hsv')
```



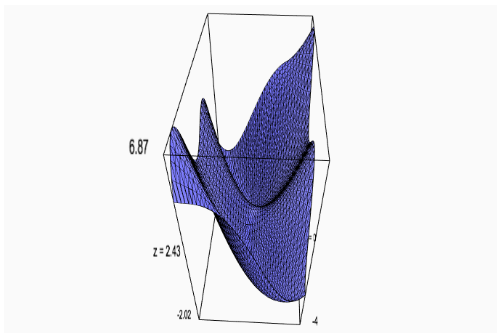

```
plot3d(sin(pi*(x^2+y^2))/2,(x,-1,1),(y,-1,1),mesh=true)
```




```
contour_plot(x^2-y*sin(x+y),(x,-2,2),(y,-4,4),plot_points=121,fill=\
    false,cmap='hsv')
```




```
plot3d(x^2-y*sin(x+y),(x,-2,2),(y,-4,4), mesh=true)
```


6.87
z = 2.43
-2.02
-4

(5) Show that
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{(x+y)^2}$$
does not exist by finding two smoothly parameterized paths to the origin which give different limits as $(0,0)$ is approached.

Solution: We approach $(0,0)$ in two different ways: by moving along the x-axis and by moving along the line $x = y$. The first case can be parameterized by $(x(t), y(t)) = (t, 0)$, which gives rise to the limit along this curve
$$\lim_{t\to 0} \frac{\sin(0)}{t^2} = 0.$$
Along the line $x = y$, we can use the parameterization $(x(t), y(t)) = (t, t)$, and so the limit along this curve of our two-variable function towards the origin is given by
$$\lim_{t\to 0} \frac{\sin\left(t^2\right)}{4t^2},$$
which is indeterminate, but by L'Hospital's rule is equal to
$$\lim_{t\to 0} \frac{\cos\left(t^2\right)}{4} = \frac{1}{4} \neq 0.$$
Since we got different answers approaching from different directions, the limit
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{(x+y)^2}$$
does not exist.