HOMEWORK 2 SOLUTIONS

MA1132: ADVANCED CALCULUS, HILARY 2017

(1) Find parametric equations for the tangent line of the graph of $r(t) = (t, \sqrt{t^2+1}, 3/t)$ when $t = 1$.

Solution: A point on this line is

$$r(1) = (1, \sqrt{2}, 3).$$

A vector parallel to this line is the tangent vector

$$r'(1) = \left(1, \frac{t}{\sqrt{t^2+1}}, -\frac{3}{t^2}\right)\Big|_{t=1} = (1, 1/\sqrt{2}, -3).$$

Thus, suitable parametric equations for the line are given by

$$\begin{cases} x = 1 + t \\ y = \sqrt{2} + \frac{t}{\sqrt{2}} \\ z = 3 - 3t. \end{cases}$$

(2) The graphs of

$$r_1(t) = (t, 2t, t^2)$$

and

$$r_2(t) = (t^2, 2t^{\frac{3}{2}}, t^6)$$

intersect when $t = 1$ at the point $(1, 2, 1)$. Find the angle between the two tangent vectors of the graphs at this point.

Solution: Recall that the angle ϑ between two vectors v_1 and v_2 can be found using the dot product via the formula:

$$\cos\vartheta = \frac{v_1 \cdot v_2}{|v_1||v_2|}.$$

In this case, the tangent vectors when $t = 1$ are given by

$$v_1 = r_1'(1) = (1, 2, 2t)|_{t=1} = (1, 2, 2),$$

$$v_2 = r_2'(1) = \left(2t, 3t^{\frac{1}{2}}, 6t^5\right)\Big|_{t=1} = (2, 3, 6).$$

Now the norms are $|v_1| = \sqrt{1^2+2^2+2^2} = 3$, $|v_2| = \sqrt{2^2+3^2+6^2} = 7$, and $v_1 \cdot v_2 = (1\cdot 2 + 2\cdot 3 + 2\cdot 6) = 20$, so that

$$\cos\vartheta = \frac{20}{21},$$

or $\vartheta = \arccos(20/21)$.

(3) (a) In single-variable calculus, you have been taught that the function $f(x) = |x|$ is not differentiable due to the sharp bend, or cusp, at the origin graph of $f(x)$. However, it is possible to find differentiable parameterizations of this curve, that is, parameterizations $r(t)$ for which $r'(t)$ exists for all t. Find one such parameterization.

(b) Just as graphs that as curves look to not be nice can have nice parameterizations, so can nice curves in the plane have not so nice parameterizations. Given any line parameterized by $r(t) = (x_0 + at, y_0 + bt, z_0 + ct)$ with $(a, b, c) \neq 0 \in \mathbb{R}^3$, note that $r'(t) = (a, b, c)$ is continuous and never vanishes, and hence that $r(t)$ is a smooth vector-valued function. Find a change of parameter $t = f(\tau)$ for which $r(f(\tau))$ parameterizes the same curve, but which does not do so smoothly.

Solution:

a). One such function is given by $r(t) = (t^3, |t^3|)$. The derivative is $r'(t) = (3t^2, 3t|t|)$. It is also useful to think about why the graph of the $f(x) = |x|^3$ is smooth (e.g., draw a picture), and I encourage you to see if you can find more suitable parameterizations.

b). We can again use the function $f(\tau) = \tau^3$, which is useful as it has an inverse function $\tau^{1/3}$, which implies that making the change of parameter $t = f(\tau)$ yields the same curve. That is, given a real number τ, clearly $r(f(\tau))$ lies on the graph of $r(t)$, as $f(\tau)$ is some real number, and any point $r(t)$ on the line is represented in the new parameterization as $r(t) = r(f(t^{1/3}))$. This is now not a smooth parameterization, though, as the chain rule implies that

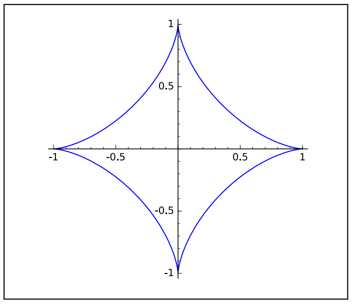
$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau},$$

and $\frac{dt}{d\tau} = 3\tau^2$, which vanishes when $\tau = 0$. Hence, $\frac{dr}{d\tau} = 0$ when $\tau = 0$.

(4) The curve parameterized by

$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$$

for $0 \leq t < 2\pi$ is known as an *astroid*, and is pictured below. Find the arclength of this astroid (hint: it is useful to use the geometric symmetries of this figure; in particular, the length of the entire curve is the same as 4 times the length of the curve from $t = 0$ to $t = \pi/2$).



Solution: The corresponding vector-valued function is given by $r(t) = (\cos^3(t)\sin^3(t))$, and we differentiate to find

$$r'(t) = (-3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)).$$

The norm of the derivative is then

$$|r'(t)| = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = 3\sqrt{\sin^2(t)\cos^2(t)(\cos^2(t) + \sin^2(t))} = 3|\sin(t)\cos(t)|.$$

As noted in the hint, we can use the obvious geometric symmetries of the figure sketched above and note that the length L we want to compute is in fact 4 times the length of the curve sketched out on the interval $t \in [0, \in \pi/2)$ (which is the section of the curve from $(1,0)$ to $(0,1)$). On this interval, $\sin(t)$ and $\cos(t)$ are both positive, and so $|\sin(t)\cos(t)| = \sin(t)\cos(t) = \frac{1}{2}\sin(2t)$. That is, on this interval,

$$|r'(t)| = \frac{3}{2}\sin(2t).$$

Thus, the length is

$$L = 4\int_0^{\frac{\pi}{2}} |r'(t)|dt = 6\int_0^{\frac{\pi}{2}} \sin(2t)dt = [-3\cos(2t)]_0^{\frac{\pi}{2}} = 6.$$

(5) Find the vector-valued functions $T(t), N(t)$, and $B(t)$ determining the unit tangent, unit normal, and binormal vectors to the helix with parameterization $r(t) = (\cos(t), \sin(t), t\sqrt{3})$.

Solution: We first take the derivative to find

$$r'(t) = (-\sin(t), \cos(t), \sqrt{3}).$$

To find $T(t)$, we simply divide $r'(t)$ by its norm, which is

$$|r'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 3} = 2.$$

That is,

$$T(t) = \frac{r'(t)}{2} = \left(-\frac{\sin(t)}{2}, \frac{\cos(t)}{2}, \frac{\sqrt{3}}{2}\right).$$

To compute the unit normal vector function, we first compute

$$T'(t) = \frac{1}{2}\left(-\cos(t), -\sin(t), 0\right).$$

The normalized version $N(t)$ is thus

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{1}{2}\left(-\cos(t), -\sin(t), 0\right)}{\frac{1}{2}\sqrt{\sin^2(t) + \cos^2(t)}} = \left(-\cos(t), -\sin(t), 0\right).$$

Finally,

$$B(t) = T(t) \times N(t) = \frac{1}{2}\left(\sqrt{3}\sin(t), -\sqrt{3}\cos(t), 1\right).$$

(6) (a) Using your answer from 5), find the curvature of the helix $r(t) = (\cos(t), \sin(t), t\sqrt{3})$ as a function of t.

(b) The curvature of a curve measures how fast it turns, or, in a sense, how much it fails to be a straight line. There is another quantity, called *torsion*, which determines to what extent a space curve fails to lie in a plane. It is defined at any point on the curve by

$$\tau = -N \cdot \frac{dB}{ds}.$$

Compute the torsion of the helix from problem 5.

Solution:

a). This may be computed using the calculations above as

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{1}{2}}{2} = \frac{1}{4}.$$

That is, at any point on the helix, the curvature is $1/4$.

b). In order to apply the definition above, it is important to take care of difference between the parameterization with respect to t and the parameterization to be with respect to arc length. We saw above that

$$|r'(t)| = 2,$$

and so the arc length s from the base point at $t = 0$ to an arbitrary point t is

$$s = \int_0^t 2du = 2t,$$

so $t = s/2$. Thus,

$$\frac{dB}{ds} = \frac{dB}{dt}\frac{dt}{ds} = \frac{1}{2}\frac{dB}{dt}.$$

Using the expression for $B(t)$ above, we see that

$$\frac{dB}{ds} = \frac{\sqrt{3}}{4}\left(\cos(t), \sin(t), 0\right).$$

Hence,

$$-N(t)\cdot\frac{dB}{ds} = -\frac{\sqrt{3}}{4}(\cos(t),\sin(t),0)\cdot(-\cos(t),-\sin(t),0) = -\frac{\sqrt{3}}{4}(-\cos^2(t)-\sin^2(t)) = \frac{\sqrt{3}}{4}.$$

Note that in this example, both the curvature and the torsion are constant functions. In fact, a theorem of Lancret states that *any* curve in $\mathbb{R}^3$ with constant ratio of curvature to torsion is a helix.