~~Algebra~~ II: Rational Canonical Form

Let F = field, $R = F[x]$, V = F-v.space of dim $n \neq \infty$,

$T: V \to V$ linear.

Then we can define ~~an~~ R-module structure on V

by $\quad x \cdot v = T(v)$ and extending to

$\sum_i c_i x^i \cdot v = \sum_i c_i T^i(v)$.

As $\dim_F(V) < \infty$, ~~$\dim$~~ $\mathrm{rk}_{F[x]}(R)$

V is f.g. as an R-module (determined by action of T on a basis)

By the fund. thm of f.g. modules / PIDs

any free R-module is a direct sum of copies of ~~F~~ $R = F[x]$ is infinite dim'l $/F$

So as $\dim_F(V) < \infty$, $\mathrm{rk}_R(V) = 0$, i.e.,

it's a torsion R-module, and

V = direct sum ~~R~~ cyclic R-modules

Fund. Thm invariant factor form $\rightsquigarrow$ rat'l canonical form of $[T]_\beta$,

elem. factors $\rightsquigarrow$ Jordan canonical form

β a basis of V,

Basic terms: $T(v) = \lambda v \Rightarrow \lambda$ = eigenvalue

v = eigenvector.

$\{v \in V \mid T(v) = \lambda v\}$ = eigenspace of λ

expanded in β-coords

Given a basis β of V, the matrix for T is $[T]_\beta$, with column $i = T(\beta_i)$

(1)

Now V is an $F[x]$ torsion module, say with $m(x)$ generating the Annihilator of V and chosen to be monic. Then $m_T(x) = m(x)$ is called the <u>minimal</u> poly of T.
Similarly, for a matrix A, ~~min~~ $m_A(x)$ is the unique monic poly. of smallest degree, $m_A(A) = 0$.

Now by the Fund. Thm, we have

$$(*)\ V \cong \bigoplus_{i=1}^{m} F[x]/(a_i(x)), \text{ where } a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x).$$

unique if required to be monic.

Now $\mathrm{Ann}(V) = (a_m(x)) \Rightarrow m_T(x) = a_m(x)$.

~~As~~ Convenient bases:

basis of $F[x]/(\underset{\text{deg. } k}{a(x)})$: $\{\bar{1}, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{k-1}\}$. reductions mod $a(x)$.

T-action is multiplication by x in $(*)$, and w.r.t. this basis, $\bar{x}^i \mapsto \begin{cases} \bar{x}^{i+1} & i = 0, 1, \ldots, k-2 \\ \bar{x}^k = -a_0 - a_1\bar{x} - \cdots - a_{k-1}\bar{x}^{k-1} & i = k-1 \end{cases}$

and $a(x) = x^k + \sum_{i=0}^{k-1} a_i x^i$

$\longrightarrow$ gives the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & & & \vdots \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix} =: \mathcal{C}_{a(x)}$$

<u>companion matrix</u>.

Doing this on each factor in $(*)$ gives the matrix for T as

$$\begin{pmatrix} \mathcal{C}_{a_1(x)} & & \\ & \ddots & \\ & & \mathcal{C}_{a_m(x)} \end{pmatrix}$$

$\leftarrow$ <u>rational canonical form</u>.

Uniqueness follows from Fund. Thm (see book)

Next time: ~~Cayley-Hamilton~~ / Computing Rat. can. form...

$\underline{\text{Th}}^{m}$: S, T lin trans. on V. T.F.A.E:

1). S, T are similar.

2). The $F[x]$-module structures on V via S, T are $\cong$.

3). S, T have the same rat$^{\underline{l}}$ canonical form.

$\underline{\text{Pf}}$: $1 \Rightarrow 2$: Say $S = UTU^{-1}$, $U \in$ ~~End~~ ~~Hom(V)~~ Isom(V).

Then $U : \underset{\text{via } T}{V} \to \underset{\text{via } S}{V}$ is also an $F[x]$-module ~~ho~~ isomorphism

e.g: $U(xv) = U(T(v)) = U(T(v)) = SU(v) = x(U(v))$.

$2 \Rightarrow 3$: ~~As~~ Let $V_1 = V$ as an $F[x]$-module via S
$V_2 = V$ " " via T.

Then $V_1 \cong V_2 \Rightarrow$ same invariant factors $\Rightarrow$ same rat$^{\underline{l}}$ can. form.

$3 \Rightarrow 1$: Then S, T have the same matrix rep$^{\underline{n}}$ w.r.t. some bases, So are the same up to change of basis $\Rightarrow$ similar $\boxed{}$

$\underline{\text{Th}}^{m}$ (Cayley-Hamilton)

The minimal poly of A divides the char poly. of A.
Moreover, char(A) = product of invariant factors of A, and char(A) divides a power of min(A)
$\Rightarrow$ their char poly, min poly have the same set of roots.

$\underline{\text{Pf}}$: Let ~~B~~ B be the rat$^{\underline{l}}$ canonical form of A.

(3)

Now $\det\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} = \prod_{i=1}^{k} \det(A_i)$ (block matrices; check Leibniz form e.g.)

here, Thus, $\mathrm{char}(B) = \prod \mathrm{char}(\mathcal{C}(a_i(x)))$

Companion mats for invariant factors a_i of A

Then one can check:

$$\mathrm{Char}(\mathcal{C}_{a(x)}) = \det\begin{pmatrix} -x & & & -a_0 \\ 1 & -x & & \vdots \\ & \ddots & \ddots & -a_{k-1} \\ & & 1 & -a_{k-1}-x \end{pmatrix} = a(x)$$

$\Rightarrow \mathrm{char}(B) = \prod a_i(x)$, Now A, B are similar,

So $\mathrm{Char}(A) = \prod a_i(x)$, as claimed.

Now $\min(A) = a_k(x)$ (largest invariant factor)

$\Rightarrow$ ~~char~~ $\min(A) \mid \mathrm{char}(A)$.

Finally, all $a_i \mid a_k$ by the defn of invariant factors,

So $\mathrm{char}(A) \mid \min(A)^k$. $\boxed{E}$

Cor: $\mathrm{char}(A)(A) = 0$; satisfies its own char poly.

Computing invariant factors:

Setup: $V = F[x]$ module with basis $\{e_1, \ldots, e_n\} = \beta$

$T: V \to V$ defined by x, $A = [T]_\beta$ matrix for T w.r.t. β.

Algorithm: 1). Use elementary row/column ops, keep track of row ops. used.

$\mathcal{B}$ given by $f_i, Tf_i, \ldots, T^{\deg(a_i(x))-1} f_i$.

5). Write the kth elt of the v-space basis of $C(U)$ in terms of the original basis $\mathcal{B}$. Use coords for k^{th} column of a matrix P. Then $P^{-1}AP$ is in rat'l canonical form, and this is the matrix for T w.r.t. the basis of $C(U)$.

[redacted]