ie we pick a good one, say

$c=1$. Then running through the above, we see that $i+\sqrt[3]{2}$ is a primitive elt.

<u>Ex:</u> Consider $\mathbb{Q}(i,\sqrt{3})$

$f(x)=x^2+1$, $g(x)=x^2-3$.

bad choices of c:

$c=\frac{i-\alpha}{\sqrt{3}-\beta}$, only choices are

$$c=\frac{i-(-i)}{\sqrt{3}-(-\sqrt{3})}=\frac{2i}{2\sqrt{3}}=\frac{i}{\sqrt{3}}=\frac{i\sqrt{3}}{3}.$$

not in $\mathbb{Q}$ any way.

Example, a primitive elt. is $i+\sqrt{3}$. Indeed,

$(i+\sqrt{3})^2=3-1+2i\sqrt{3}=2+2i\sqrt{3}$,

$(i+\sqrt{3})^4=(2+2i\sqrt{3})^2=-8+8i\sqrt{3}$

$\Rightarrow (i+\sqrt{3})^4-4(i+\sqrt{3})^2=-16$

$\Rightarrow$ its a root of x^4-4x^2+16. Check its the min poly $\rightsquigarrow$ degree is $4 \Rightarrow \mathbb{Q}(i,\sqrt{3})=\mathbb{Q}(i+\sqrt{3})$.

<u>Ex</u>: ~~Roots~~ Reduction mod primes fails...

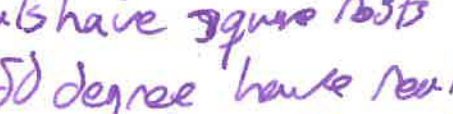
But! But look at roots in $\mathbb{C}$: $\{\pm\sqrt{\pm 2i\sqrt{3}+2}\}$

If factored, would factor into product of quadratics, over $\mathbb{Q}\subseteq\mathbb{R}$ both prods. of over complex conj. pairs, multiply out,

[redacted]

Fundamental Theorem of Algebra:

$\mathbb{C}$ is algebraically closed.

Pf: $\mathbb{C} = \mathrm{Split}_{\mathbb{R}}(x^2+1) = \mathbb{R}[i]$. Still need topology!
Intermediate value theorem $\Rightarrow$ positive reals have square roots (picture:  $x^2-a=0$) and real polys of odd degree have real roots (picture: )

Claim: Every elt. in $\mathbb{C}$ has a square root.
Pf: Let $\alpha \in \mathbb{C}$, say $\alpha = a+bi$, $a,b \in \mathbb{R}$. Let $c, d \in \mathbb{R}$ be s.t.

$c^2 = \frac{a+\sqrt{a^2+b^2}}{2}$, $d^2 = \frac{-a+\sqrt{a^2+b^3}}{2}$, so that $c^2 - d^2 = a$

and $(2cd)^2 = b^2$. Pick signs of c, d so that $\mathrm{sgn}(cd) = \mathrm{sgn}(b)$, and then we have $(c+di)^2 = (c^2-d^2) + 2cd\,i = a+bi = \alpha$.

Now given any $f(x) \in \mathbb{R}[x]$ (rel. prime to x^2+1), WNTS if $E = \mathrm{Spl}_{\mathbb{R}}(f(x)(x^2+1))$, then $E = \mathbb{C}$. $\mathrm{char}(\mathbb{R}) = 0 \Rightarrow \mathbb{R}$ perfect $\Rightarrow f(x)(x^2+1)$ separable $\Rightarrow E/\mathbb{R}$ Galois. Let $G := \mathrm{Gal}(E/\mathbb{R})$, $H \leq G$ a 2-Sylow subgp. Consider the fixed field $M := E^H$. Then $[M:\mathbb{R}] = [G:H]$ is odd ~~...~~ By the primitive (α) elt. theorem, $M = \mathbb{R}(\alpha)$ ...

Now $\text{minpoly}_{\mathbb{R}}(\alpha)$ has degree $[M:\mathbb{R}]$ which is odd

$\Rightarrow$ $\text{min poly}_{\mathbb{R}}(\alpha)$ has a root in $\mathbb{R}$ $\Rightarrow$ $\text{min poly}_{\mathbb{R}}(\alpha) = x - \alpha$ (its irreducible) (degree 1)

$\Rightarrow$ $M = \mathbb{R}$ and so $G = H$. Thus, G is a 2-gp.

Thus, $\text{Gal}(E/\mathbb{C}) =: G'$ is a 2-gp.

Suppose $G' \neq 1$. Then as G' is a 2-gp, $\exists\, N \leq G'$ of index 2 and the fixed field E^N has $[E^N : \mathbb{C}] = [G' : N] = 2$.

But all field extensions of degree 2 (we showed!) are obtained by taking square roots, but all square roots of elts. in $\mathbb{C}$ are in $\mathbb{C}$ by above. Thus, $G' = 1$

$\Rightarrow E = \mathbb{C}$

$\Rightarrow \mathbb{C} = \overline{\mathbb{C}}$. $\boxtimes$

[redacted]