Goal: We aim to show that in a tower $E/K/F$,

E/K sep. + K/F sep. $\Rightarrow$ E/F sep.

The "natural" way to study this is via embeddings.

Def^{n} E/F finite. The separable degree, $[E:F]_s$, is the number of distinct embeddings $E \hookrightarrow \bar{F}$ fixing F.

Th^{m}: $E \supseteq K \supseteq F$, E/F finite. Then.

$$[E:F]_s = [E:K]_s \cdot [K:F]_s.$$

Pf: Set $[K:F]_s =: m$, $[E:K]_s =: n$.

Let $\varphi_1, \dots, \varphi_m$ be the embeddings of $K \hookrightarrow \bar{F}$ over F,

$\psi_1, \dots, \psi_n$ the embeddings of $E \hookrightarrow \bar{F}$ over K.

Let Ω_0 be a splitting field over F containing E.

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, dash] & \bar{F} \\ K \arrow[r, hook, "\varphi_i"] \arrow[d, dash] & \bar{F} \\ F & \end{tikzcd}$$

Fix a φ_i.

Claim: The number of embeddings τ of E into $\bar{F}$ over F restricting to φ_i over K is n.

Pf: Use our ext^{n} theorem (13.4, Th^{m} 27 in D-F) to extend φ_i to $\tilde{\varphi}_i : \Omega_0 \to \Omega_0$.

This is possible as ~~Ω_0~~ Ω_0 is a splitting field of $f \in F[x]$ so it's also a splitting field of f over K and $\varphi_i(K)$.

Clearly, $\tilde{\varphi}_i : \Omega_0 \hookrightarrow \Omega_0$ is injective.

Look at ~~Define~~ $\phi := \tilde{\varphi}_i^{-1} \tau : E \to \bar{F}$.

Now $\Phi|_K = id_K$: if $x \in K$, $\tau(x) = \varphi_i(x)$

by assumption $\Rightarrow$ ~~$\tilde{\varphi}_i(\tau(x)) = \varphi_0$~~

$\Rightarrow \tilde{\varphi}_i^{-1}(\tau(x)) = \varphi_i^{-1}(\varphi_i(x)) = x$.

Thus, Φ is an embedding of E fixing K

$\Rightarrow \Phi = \psi_j$, for some j.

Therefore, $\tau = \tilde{\varphi}_i \psi_j$ for some j,

So there are exactly n possible τ. $\square$

We now count embeddings $\tau : E \hookrightarrow \overline{F}$:

For each $i = 1, \dots, m$, there are exactly n extending φ_i, giving mn total.

Thus, $[E:F]_s = mn = [E:K]_s \cdot [K:F]_s$. $\square$

<u>Theorem 2</u>: ~~If~~ $E = F(\alpha)$, $\alpha \in E$ a root of an irred. poly. $f \in F[x]$. Then $[E:F]_s$ is the # of distinct roots of f.

Since $[E:F] = \deg(f)$, we have

$[E:F]_s \leq [E:F]$, with equality iff. f is separable

<u>Pf</u>: Assume WLOG f is monic. Let $n := \deg(f)$,

Write $f(x) = \prod_{i=1}^{n} (x - \alpha_i)$.

By Thm 6 of D-F, §13.1, $\forall i$, $\exists$ embedding

$\varphi_i : F(\alpha) \hookrightarrow \overline{F}$. with $\varphi_i(\alpha) = \alpha_i$.

Moreover, if $\varphi: F(\alpha) \hookrightarrow \bar{F}$ is any embedding, it sends α to some α_i, so $\varphi = \varphi_i$ for some i. Also, $\varphi_i = \varphi_j \Leftrightarrow \alpha_i = \alpha_j$. Thus, $[E:F]_s = \#$ distinct φ_i $= \#$ distinct α_j ∎

<u>Thm 3</u>: For any finite extension E/F,

$$[E:F]_s \leq [E:F].$$

<u>Pf</u>: We just did the case of a simple extension in Thm 2.

Generally: say $E = F(\alpha_1, \ldots, \alpha_n)$. Do a tower of simple extns $F \subseteq F(\alpha_1) \subseteq F(\alpha_1, \alpha_2) \subseteq \cdots \subseteq E$

Now each $[F(\alpha_1, \ldots, \alpha_k) : F(\alpha_1, \ldots, \alpha_{k-1})]_s \leq [F(\alpha_1, \ldots, \alpha_k) : F(\alpha_1, \ldots, \alpha_{k-1})]$

Multiplying these inequalities gives the result ∎

Finally, we ~~obtain~~: define E/F is <u>separable</u> $\Leftrightarrow [E:F]_s = [E:F]$.

<u>Thm 4</u>: $E \supseteq K \supseteq F$ finite extn. Then

$$E/F \text{ sep.} \Leftrightarrow E/K \text{ and } K/F \text{ sep.}$$

<u>Pf</u>: We write $[E:K]_s [K:F]_s = [E:F]_s \leq [E:F] = [E:K] \cdot [K:F]$

Thus, $[E:F]_s = [E:F] \Leftrightarrow$ both $[E:K]_s = [E:K]$ and $[K:F]_s = [K:F]$ ∎

For completeness, we connect to the books def$^{\underline{n}}$ of separable:

$\underline{\text{Th}^{\underline{m}}}$ E/F finite. Then E/F is sep. (@ above def$^{\underline{n}}$) iff every $\alpha \in E$ is a root of a sep. poly in $F[\underline{x}]$.

$\underline{\text{Pf}}$: If E/F sep., ~~by~~ Th$^{\underline{m}}$ 4

$\Rightarrow F(\alpha)/F$ is sep. $\forall \alpha \in E$, and Th$^{\underline{m}}$ 2

$\Rightarrow \alpha$ is a root of a separable poly.

$\underline{\text{Conversely}}$, suppose all $\alpha \in E$ are roots of sep. polys. To show E/F is separable, we induct on $\underline{\text{degree}}$ $n := [E:F]$.

If $n=1$, $E=F$, and its clear.

If $n>1$, assume true $\forall$ ext$^{\underline{ns}}$ of degree $<n$.

For ~~$\alpha \in E$~~ $\alpha \in E \setminus F$, let $K = F(\alpha)$.

As α is a root of a sep. poly./F, Th$^{\underline{m}}$ 2

$\Rightarrow F(\alpha)/F$ is sep. To use Th$^{\underline{m}}$ 4, we have to show E/K is sep. Every elt of E is a root of a sep. poly /F, hence over K too.

As $[E:K] < [E:F]$, by the induction hypothesis, E/K is separable, as required. $\blacksquare$