# Algebra II Modules/PIDs [redacted]

Def$^{n}$:

$M$ is Noetherian if every chain

$M_1 \leq M_2 \leq M_3 \leq \cdots$ terminates

$M_k = M_{k+1} = \cdots$ some $k$.

A ring $R$ is Noetherian if it is as a $R$-module (i.e., chains of ideals terminate).

Th$^{m}$: T.F.A.E.

1). $M$ is a Noetherian module

2). Every nonempty set of submodules of $M$ has a max$^{\underline{al}}$ elt. (under containment)

3). Submodules of $M$ are all finitely generated.

Pf: $1 \Rightarrow 2$: If ~~S~~ $S \neq \emptyset$ is a set of submodules, pick any $M_1 \in S$. If max$^{\underline{al}}$, done. If not, pick $M_1 < M_2$. If $M_2$ max$^{\underline{al}}$ done. If no maximal elts ~~If~~ By Axiom of choice, can build a chain $M_1 < M_2 < \cdots$ never terminating, contradicting (1).

$2 \Rightarrow 3$: Let ~~S =~~ $N \leq M$ and $S := \{$f.g. submodules of $N\}$

$\{0\} \in S \Rightarrow S \neq \emptyset$. Thus, by (2), $S$ has a max$^{\underline{al}}$ elt. $N'$. If $N' \neq N$, let $x \in N \setminus N'$. As $N' \in S$

$N'$ is f.g. But then the submodule gen. by $N'$ and $x$ is f.g. Contradicts maximality of $N$.
So $N = N'$ is f.g.

$3 \Rightarrow 1$: Let $M_1 \leq M_2 \leq \cdots$ be a chain $\leq$ of submodules

Take $N := \bigcup_{i \geq 1} M_i$, which is a submodule.

$(3) \Rightarrow N$ is ~~finite~~ f.g., say gen's are $a_1, \ldots, a_n$.

Then $a_i \in N \; \forall i \Rightarrow$ each $a_i \in M_{j_i}$, some $j_i \in \mathbb{N}$.

Then if $m = \max\{j_1, \ldots, j_n\}$, $a_i \in M_i \; \forall i$

$\Rightarrow \text{span}(\{a_i\}) = N \subseteq M_m$. Thus, $M_m = M_{m+1} = \cdots = N$, stabilizes. $\square$

Cor: If $R$ is a PID, then $R$ is Noetherian

Pf: Ideals are f.g. as... simply gen!

N.B: We kind of used this before — things have prime factorizations

Prop: $R$ = domain, $M$ = free $R$-module of rk $n < \infty$.

Then any $n+1$ elts of $M$ are lin. dep.

Pf: Descend to fields $R \hookrightarrow F$ field of fractions.

$M \cong R^n \Rightarrow M \subseteq F^n$. $n+1$ elts in $F^n$ are lin. dep. over $F$. Say given $x_1, \ldots, x_{n+1} \in M$, $\sum_i c_i x_i = 0$, $c_i \in F$ not all 0. Then clear denominators $\rightsquigarrow \sum_i (c_i D) x_i = 0$, $c_i D \in R$.
(or)

Recall: torsion: $\mathrm{Tor}(M) := \{x \in M \mid rx = 0$ some $r \in R \setminus \{0\}\}$

Annihilators: $N \leq M$ $\mathrm{Ann}(N) := \{r \in R \mid rn = 0\ \forall n \in N\}$

<u>Def</u> For a domain $R$, the <u>rank</u> of an $R$-module $M$ is the maximum # of $R$-lin. ind. elts of $M$.

<u>Th</u>: $R$ is a PID. $M$ = free $R$-module of rk $n < \infty$. $N \leq M$.

1). $N$ is free of rk $\leq$ (= m) $n$.

2). $\exists$ basis $\{x_1, \ldots, x_n\}$ of $M$ where $\{a_1 x_1, \ldots, a_m x_m\}$ is a basis of $N$ and $a_1, \ldots, a_m \in R \setminus \{0\}$ with $a_1 | a_2 | \cdots | a_m$.

<u>Pf</u>: Nothing to say if $N = 0$. Say $N \neq 0$.

Given $\varphi \in \mathrm{Hom}(M, R)$, $\varphi(N) \leq N \Rightarrow \varphi(N)$ is an ideal in $R$

As it's a PID, $\varphi(N) = (a_\varphi)$, some $a_\varphi \in R$

Let $S := \{(a_\varphi) \mid \varphi \in \mathrm{Hom}(M, R)\}$.

$S \neq \emptyset$, as if $\varphi = 0$, then $(a_\varphi) = (0) \in S$.

By the Cor, $S$ has a max<sup></sup>l elt. Say it's $(a_\nu) = \nu(N)$.

Let $a_1 = a_\nu$, and pick a preimage $y \in N$ @ $\nu(y) = a_1$.

<u>Claim</u>: $a_1 \neq 0$. Indeed let $\{x_1, \ldots, x_n\}$ be a basis of $M$

Let $\pi_i : M \twoheadrightarrow R$ be projection onto $i$-th coord

$N \neq 0 \Rightarrow \exists\, i$ @ wh $\pi_i(N) \neq 0$. Thus,
$S$ contains more than just $(0)$. As $(a_1)$ is max'l, $a_1 \neq 0$.

Claim: $a_1 \mid \varphi(y)\ \forall\ \varphi \in \mathrm{Hom}_R(M,R)$.

Indeed let $(a_1, \varphi(y)) = (d)$ (as ideals in $R$)

$\Rightarrow \exists\, r_1, r_2 \in R,\ d = r_1 a_1 + r_2 \varphi(y)$

Let $\psi : M \to R$, $\psi = r_1 \nu + r_2 \varphi \Rightarrow \psi(y) = d$.

$\Rightarrow d \in \psi(N) \Rightarrow (d) \subseteq R\psi(N)$. Now $d \mid a_1 \Rightarrow (a_1) \subseteq (d)$ and $\psi(N) \subseteq (a_1)$ by maximality

$\Rightarrow (a_1) = (d) = \psi(N)$ by maximality of $(a_1)$.

In particular, $(a_1) = (d) \Rightarrow a_1 \mid \varphi(y)$ since $d \mid \varphi(y)$.

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In particular, applying the claim to the $\pi_i$, to here
$a_1 \mid \pi_i(y)\ \forall i$. Say $\pi_i(y) = a_1 b_i$, some $b_i \in R$.

Let $y_1 := \sum_{i=1}^{n} b_i x_i$. e.g.

Now $a_1 y_1 = \sum_i a_1 b_i x_i = \sum_i \pi_i(y) x_i = y$. (basis elts)

Directness of sum special case(s)

Then $a_1 = \nu(y) = \nu(a_1 y_1) = a_1 \nu(y_1)$ and $a_1 \neq 0$ in the domain $R$

$\Rightarrow \nu(y_1) = 1$.

We claim $y_1$ can be one basis elt of $M$ and $a_1 y_1$ can be chosen as a basis elt of $N$.

That is, specifically, a) $M = R y_1 \oplus \ker(\nu)$, b) $N = R a_1 y_1 \oplus (N \cap \ker \nu)$.

a): For $x \in M$, let $x = \nu(x) y_1 + (x - \nu(x) y_1)$ ~~$(x - \nu(x))$~~

Continued next time

$\Rightarrow \nu(x - \nu(x) y_1) = \nu(x) - \nu(x)\nu(y_1) = \nu(x) - \nu(x)$

$= 0 \Rightarrow x - \nu(x) y_1 \in \ker(\nu) \Rightarrow x \in R y_1 + \ker(\nu)$.

Suppose $r y_1 \in \ker(\nu)$. Then $0 = \nu(r y_1) = r \nu(y_1) = r \Rightarrow$ this elt is $0 \Rightarrow R y_1 \cap \ker(\nu) = 0$.

For (b), note $a_1 \mid \nu(x')\ \forall x' \in N$, as $a_1$ is a gen of $\nu(N)$.

Say $\nu(x') = b a_1$, then by (a), $x' = \nu(x') y_1 + (x' - \nu(x') y_1)$ $= b a_1 y_1 + (x' - b a_1 y_1) \in R a_1 y_1 + \ldots$