Algebra II: Modules / PIDs: Fund. Th$^{\underline{m}}$ ~~Existence~~ Uniqueness.

(an gp. to get the p-parts... e.g.: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30}$
$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
$\cong (\mathbb{Z}_2^5) \oplus (\mathbb{Z}_3^2) \oplus \mathbb{Z}_5$)

Th$^{\underline{m}}$. (Primary decomposition)

$R =$ PID, $M =$ torsion R-module, not 0.
a is the annihilator of M, with prime factorization

$$a = u p_1^{\alpha_1} \cdots p_n^{\alpha_n}, \quad u \in R^*, \ p_i \text{ distinct}$$

For $i = 1, 2, \ldots, n$, let $N_i := \{x \in M \mid p_i^{\alpha_i} x = 0\}$.
Then $N_i \leq M$ with annihilator $p_i^{\alpha_i}$, and
$N_i = \{x \in M \mid p_i^n x = 0 \text{ some } n \geq 0\}$.

Moreover, $M = N_1 \oplus N_2 \oplus \cdots \oplus N_n$.

If, further, M is f.g., then
N_i is the direct sum of a finite # of cyclic modules w/ annihilators dividing $p_i^{\alpha_i}$.
The N_i are called the p_i-primary components of M.

Pf. The case when N is f.g. follows from the theorem above. More generally, clearly
$N_i \leq M$, ann dividing $p_i^{\alpha_i}$.
$R =$ PID $\Rightarrow (p_i^{\alpha_i}), (p_j^{\alpha_j})$ comax$^{\underline{\text{al}}}$, $i \neq j$,
$\rightsquigarrow$ direct sum decomp. is shown via the module version of the C.R.T.

This decomp. shows $\mathrm{Ann}(N_i) = p_i^{\alpha_i}$.

For example, the elem. divisors of a f.g. M are the invariant factors of primary comps of $Tor(M)$!

Ex: As above, take primary decomp: $(\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_3)^3 \oplus \mathbb{Z}_5$. Invariant factors are $2,2,2,2,2 \mid 3,3,3 \mid 5$

Lemma: $R = PID$, $p \in R$ prime.

$F := R/(p)$ is a field.

Then 1) if $M := R^r$, 1) $M/pM \cong F^r$,

2) If $M = R/(a)$, $a \in R \setminus \{0\}$, then

$$M/pM \cong \begin{cases} F & p \mid a, \\ 0 & p \nmid a. \end{cases}$$

3) Let $M = \bigoplus_{i=1}^{k} R/(a_i)$, where $p \mid a_i \ \forall i$.

Then $M/pM \cong F^k$.

Pf: 1). Projection:

$$\pi: R^r \twoheadrightarrow (R/(p))^r$$
$$(\alpha_i) \mapsto (\alpha_i \bmod p)$$

$\ker(\pi) = \{(p\alpha_i) \mid \alpha_i \in R\} = \{p(\alpha_i) \mid \alpha_i \in R\} = pR^r$

$\underset{1^{st}\text{ Iso}}{\Rightarrow} R^r/pR^r \cong R/(p)^r = F^r$.

2). Projection $\pi: R \to R/(a)$
$$r \mapsto r \bmod a$$

$\pi((p)) = p(R/(a)) = [(p)+(a)]/(a)$

Now $(p)+(a) = (\gcd(p,a)) = \begin{cases} (p) & p \mid a \\ R = (1) & p \nmid a \end{cases} \Rightarrow \pi(p) = \begin{cases} (p)/(a) & p \mid a \\ R/(a) & p \nmid a \end{cases}$

Thus, $M/pM = \begin{cases} R/(a) \,/\, ((p)/(a)) \cong R/(p) = F, & p \mid a, \\ M/M = 0, & p \nmid a. \end{cases}$

31. Follows from (1) just as in proof of earlier Th^{m} (Fund. Th^{m}, uniqueness) (invariant existence th^{m}).

Th^{m} 1). M_1 & M_2 f.g. R-modules.

Then $M_1 \cong M_2 \iff rk(M_1) = rk(M_2)$ and same invariant factors.

2). $M_1 \cong M_2 \iff rk(M_1) = rk(M_2)$ and same elementary divisors.

Pf. In both cases, $\Leftarrow$ is clear.

Suppose then that $\underset{(\text{rank } r_1)}{M_1} \cong \underset{(\text{rank } r_2)}{M_2}$. If $\varphi : M_1 \xrightarrow{\sim} M_2$, then $\varphi(Tor(M_1)) \subseteq Tor(M_2)$ and vice-versa, (isomorphism descends) So $Tor(M_1) \cong Tor(M_2)$. Thus,

$R^{r_1} \cong M_1/Tor(M_1) \cong M_2/Tor(M_2) \cong R^{r_2}$

We showed before that then $r_1 = r_2$; also follows here by applying (1) of the Lemma.

We are now reduced to torsion modules, so $r_1 = r_2 = 0$.

For elementary divisors, it's enough to show for any prime p, the elem. divisors which are a power of p are the same for M_1, M_2.

$M_1 \cong M_2 \Rightarrow p$-primary comp.$(M_1) \cong p$-primary comp.$(M_2)$

as these are the submodules of elts. ann. by powers of p.

Thus, WNTS only that if M_1, M_2 have annihilator a power of p and $M_1 \cong M_2$ then they have the same elem. divisors.

Induction on power of p in $Ann(M_1)$:

Base case: p^0: Then $M_1 = M_2 = 0$ ✓

Inductive step: $p^{>0}$:

Then M_1, M_2 have non-trivial elem divisors.

Say M_1 has elem divisors $\underbrace{p, \dots, p}_{m \text{ times}}, p^{\alpha_1}, \dots, p^{\alpha_s}$

with $2 \le \alpha_1 \le \dots \le \alpha_s$.

Then pM_1 has elem divisors

$$p^{\alpha_1 - 1}, p^{\alpha_2 - 1}, \dots, p^{\alpha_s - 1}$$

Similarly, for M_2, if elem divisors are $\underbrace{p, \dots, p}_{n \text{ times}}, p^{\beta_1}, \dots, p^{\beta_t}$

then pM_2 has elem divisors $p^{\beta_1 - 1}, \dots, p^{\beta_t - 1}$

Then as $M_1 \cong M_2$ and $pM_1 \cong pM_2$, and the power of p in $Ann(pM_1)$ is one less than the power of p in $Ann(M_1)$.

induction $\Rightarrow$

elem. divisors of pM_1 = elem. divisors of pM_2.

Thus, $s=t$, $\alpha_i - 1 = \beta_i - 1$, $i = 1, \dots, s$

$\Rightarrow \alpha_i = \beta_i \ \forall i$.

Then, as $M_1/pM_1 \cong M_2/pM_2$, by (3) of the Lemma,

$F^{m+s} \cong F^{n+t}$ $\Rightarrow m+s = n+t \Rightarrow m = n$.

For invariant factors, say, those for M_1 are

$a_1 | a_2 | \cdots | a_m$.

The elem. divisors are obtained by taking the prime power factors of these elts. (as ~~above~~ last time).

The divisibility rel$^{\underline{ns}}$ $\Rightarrow$ a_m = product of largest of the prime powers among these elem. divisors,

a_{m-1} = prod of largest prime powers among the elem. divisors once factors for a_m are removed, etc.

Thus, the invariant factors are uniquely determined by the elem. divisors, which are unique, and, hence, so are the invariant factors.

Cor! (Fund. Th$^{\underline{m}}$ of f-g abelian gps)

Pf: Let $R = \mathbb{Z}$. $\blacksquare$

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