Algebra I: Algebraic closures, seperable extns.

Defn: If K/F is algebraic, is the splitting field for a set of polys in F, then K is a normal extn of F.

Defn: An algebraic ~~ate~~ closure of F is a field extn $\overline{F}/F$ algebraic s.t. every poly. in F splits completely over $\overline{F}$.
(and so contains every algebraic elt of F)
F is called algebraically closed if ~~$\overline{F}=F$~~, every poly. in F has a root in F (which automatically $\Rightarrow$ every poly splits)

Prop: Any closure $\overline{F}$ is ... closed! ($\overline{\overline{F}}=\overline{F}$) generally what one means by "closure"

Pf: If $f(x) \in \overline{F}[x]$, and α a root of $f(x)$, then consider $\overline{F}(\alpha)/\overline{F}$. This is algebraic, and so is $\overline{F}/F$ and so $\overline{F}(\alpha)/F$ alg. $\Rightarrow \alpha$ alg. $/F \Rightarrow \alpha \in \overline{F}$.

Prop: Every field F lies in an alg. closed field K.

Pf: Let $f \in F[x]$, nonconstant, monic.
Associated to this, assign a variable x_f.
Consider $F[\{x_f\} \mid f$ is a nonconstant monic$]$. Big poly ring.
$I :=$ ideal generated by each $f(x_f)$.
We claim that I is proper. Suppose not.
Then $1 \in I \Rightarrow \exists$ reln $g_1 f_1(x_{f_1}) + \dots + g_n f_n(x_{f_n}) = 1$
(1) (*)

with ~~&~~ g_i polys in the x_f.

Let $x_{f_i} = x_i$, and $x_{n+1}, \dots, x_m$ the rest of the vars in all the g_is.

Then (*) becomes

$$g_1(x_1,\dots,x_m) f_1(x_1) + \dots + g_n(x_1,\dots,x_m) f_n(x_n) = 1$$

Take F' to be a finite extension of F containing a root α_i of each f_i.

~~Let~~ Setting $x_i = \alpha_i \ \forall i=1,\dots,n$ and $x_{n+1} = \dots = x_m = 0$ above gives $0=1$ in F' so ✱.

Thus, I really is proper. Thus, by Zorn's Lemma, $I \subseteq M$, M a max$^{\underline{ml}}$ ideal.

Thus $k_1 := F[\{x_f\}]/M$ is a field containing a copy of F. Now $\forall f$, $f(x_f) \in I \subseteq M$

~~Now $\forall i=1,\dots,n$, $f(x$~~. So f has a root in k_1.

Thus, k_1 is a field where every poly $/F$ has a root.

Now we can build a field k_2/k_1 in the same way where all polys $/k_1$ have a root. Continuing gives a chain

$$F = k_0 \subseteq k_1 \subseteq \dots \subseteq k_n \subseteq \dots$$

where $f(x) \in k_j[x]$ (nonconstant) $\Rightarrow f(x)$ has a root in k_{j+1}.

Set $K := \bigcup k_j \supseteq F$. Any poly in K has ~~roots~~ coeffs contained in some k_n $\Rightarrow$ root in $k_{n+1} \subseteq K$.

f: If F is a field and K an alg. closed ext^n, then the set $\bar{F}$ of elts in K which are algebraic/F is an ~~algly~~ closure of F. It is unique up to $\cong$.

Pf: By def^n, $\bar{F}$ is algebraic $/F$.

Every poly $f \in F[x]$ splits over K.

But each root α of f is algebraic $/F$ and so is in $\bar{F}$. Thus f actually splits over $\bar{F}$, so $\bar{F}$ is a closure of F.

Uniqueness is the same as for splitting fields (+ Zorn). ☒

Th^m (Fund. Thm of Algebra)

$\mathbb{C}$ is closed.

Def^n: A poly $/F$ is separable if all roots are simple (in a splitting field, say).

If a poly ~~is not~~ has a mult. root, it's called inseparable.

Lemma: f is separable $\iff$ $(f(x), f'(x)) = 1$

Pf: f has a mult. root $\iff$ ~~$(x-\alpha) \mid f(x), f'(x)$~~

$f(\alpha) = f'(\alpha) = 0 \iff (x-\alpha) \mid f(x), f'(x) \iff (x-\alpha) \mid (f(x), f'(x))$

Lemma: Irred. polys over fields of char 0 are sep.
More generally, f / char 0 is sep. $\Leftrightarrow$ prod. of distinct irreducibles.

Pf: F char$(F)=0$, $f(x)$ is irred. of deg. n.
Then $\deg(f')=n-1$. But f irred. $\Rightarrow$ only factors are deg. 0, n $\Rightarrow$ $(f,f')=1$.
The second part holds since distinct irreducibles can't have roots in common (think about minimal polys).

Def: F is perfect if char$(F)=0$ or char$(F)=p$ and Frob$_p$ is ~~inject~~ surjective. (e.g. for finite fields)

Prop: Irred. polys / perfect fields are sep.
Pf: We did char. 0. Suppose char$(F)=p$, f irred. / F. If ~~the~~ f was inseparable, then the same proof as above for char ~~p~~ 0 would work, but the degree of $f'(x)$ may not be $n-1$, e.g.,
$(x^p)' = px^{p-1}=0$. has deg. 0.
Clearly, $f'(x)=0 \Leftrightarrow$ every exponent is a mult of p.
that is, $f(x) = a_n x^{np} + \dots + a_0 = (a_n x^n + \dots + a_0)|_{x=x^p}$
So $f(x)=g(x^p)$ some $g(x)$.
But if F is perfect, as each $a_i = b_i^p$ some b_i

But then:

$$f(x) = g(x^p) = \sum_j a_j x^{jp} = \sum_j (b_j x^j)^p = \left(\sum_j b_j x^j\right)^p$$

$\Rightarrow f(x)$ is a pth-power $\Rightarrow f(x)$ wasn't irreducible $\Rightarrow$ contradiction $\boxed{}$

But if $f(x)$ is inseparable irreducible, then $f'(x) = 0 \Rightarrow f(x) = f_1(x^p)$.

If separable, good. If not, then ~~for~~ $f_1(x) = f_2(x^p)$ some f_2, ... and eventually, we get $f(x) = f_k(x^{p^k})$, $(f_k(x))' \neq 0$. Smallest such k. Thus, f_k is ~~eas~~ easily shown to be irred. $\Rightarrow$

<u>Prop</u>: If $f(x)$ irred. $/F$ of char p, $\exists!$ $k \geq 0$ and a unique irred. sep. poly f_{sep} $/F$ s.t.

$$f(x) = f_{sep}(x^{p^k}).$$

We call ~~p_{sep} the~~ $\deg(p_{sep})$ the <u>separable degree</u> of f, called $\deg_s(f)$. p^k is the <u>inseparable</u> degree, denoted $\deg_i(f)$. Thus, $\deg(f) = \deg_s(f) \cdot \deg_i(f)$.

Finally, <u>Def</u>: K/F is <u>separable</u> if every elt. of K is the root of a sep. poly $/F$ ($\Leftrightarrow$ min polys of all elts sep.)

~~Ex~~ Cor: Finite ext(s) of perfect fields are always sep.