Modules: Jordan Canonical Form:

Same as before, but for elementary divisors.
Nicer, but over algebraic closures.

invariant factors: $a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x)$

elem. divisors: the powers of irred. factors of inv. factors
"primes"

Nicest case: The $a_i(x)$ factor into linear terms.

As $\prod a_i(x) = \mathrm{char}(T)$, that each poly. splits over K
$\iff$ eigenvalues of $T \subseteq K$.

In this case, each elem. divisor is of form
$$(x-\lambda)^k.$$

Thus, get decomp. $V \cong \bigoplus F[x]/(x-\lambda)^k$.

Basis of ~~term~~ factr.: $\bar{1}, \overline{x-\lambda}, \overline{(x-\lambda)^2}, \ldots, \overline{(x-\lambda)^{k-1}}$.
but write in backwards order.

These are a basis, as the matrix for these
in terms of std. basis $\bar{1}, \bar{x}, \overline{x^2}, \ldots, \overline{x^{k-1}}$

is $\begin{pmatrix} 1 & 1 & 1 & \\ 0 & -\lambda & -2\lambda & \\ \vdots & 0 & \lambda^2 & \\ 0 & \vdots & 0 & \cdots \\ & & \vdots & \end{pmatrix}$ is upper Δ with no 0 entries on diag. $\Rightarrow \det \neq 0$.

(1)

Now in terms of this basis, applying T (mult. by $\bar{x}$),

$$\rightsquigarrow \quad \overline{(x-\lambda)^{k-1}} \longmapsto x(x-\lambda)^{k-1} = ((x-\lambda)+\lambda)(x-\lambda)^{k-1} = \lambda(x-\lambda)^{k-1} + (x-\lambda)^{k} = \lambda(x-\lambda)^{k-1}$$

$$\overline{(x-\lambda)^{k-2}} \longmapsto \overline{x(x-\lambda)^{k-2}} = \cancel{\lambda(x-\lambda)^{k-2}} + (x-\lambda)^{k-1}$$

$$= ((x-\lambda)+\lambda)(x-\lambda)^{k-2} = (x-\lambda)^{k-1} + \lambda(x-\lambda)^{k-2}$$

$$\vdots$$

$$1 \longmapsto \cancel{x \cdot 1}\ \bar{x} = \lambda + (\bar{x}-\lambda)$$

$$\rightsquigarrow \text{matrix} \quad \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

Jordan block size k eigenvalue λ.

Jordan Canonical Form:

$$[T] = A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}, \quad J_i \text{ Jordan blocks.}$$

Note: If each eigenvalue has mult. 1, i.e. $\min(A)$, $\min(A)$ has no repeated roots, then each Jordan block is just a λ, so $JCF(A) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow A$ is diagonalizable. Conversely, if A is d'able then its in a Jordan form, all roots of $\min(A)$ simple.

More generally, Jordan canonical form is the closest one can get to d'able matrices.

xamples: (Book gives whole long num. alg.)

Given a trans, pick a basis & compute matrix. So assume we have a matrix already.

Seems almost everyone skips it!

$$A = \begin{pmatrix} -1 & 3 & -1 & 0 & -2 & 0 & 0 & -2 \\ -1 & -1 & 1 & 1 & -2 & -1 & 0 & -1 \\ -2 & -6 & 4 & 3 & -8 & -4 & -2 & 1 \\ -1 & 8 & -3 & -1 & 5 & 2 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 \end{pmatrix}$$

Stick with book one!!

$$A = \cancel{\begin{pmatrix} 0 & 0 & 12 \\ 1 & 0 & -16 \\ 0 & 1 & 7 \end{pmatrix}} \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$$

$$xI - A = \begin{pmatrix} x-2 & 2 & -14 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix}$$

$$\xrightarrow[R_1+R_2]{} \begin{pmatrix} x-2 & x-1 & -7 \\ 0 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \xrightarrow[C_1-C_2]{} \begin{pmatrix} -1 & x-1 & -7 \\ -x+3 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix}$$

$$\xrightarrow[-R_1]{} \begin{pmatrix} 1 & 1-x & 7 \\ -x+3 & x-3 & 7 \\ 0 & 0 & x-2 \end{pmatrix} \xrightarrow[R_2+(x-3)R_1]{} \begin{pmatrix} 1 & 1-x & 7 \\ 0 & -x^2+5x-6 & 7(x-2) \\ 0 & 0 & x-2 \end{pmatrix}$$

$$\xrightarrow[C_2+(x-1)C_1]{} \begin{pmatrix} 1 & 0 & 7 \\ 0 & -x^2+5x-6 & 7(x-2) \\ 0 & 0 & x-2 \end{pmatrix} \xrightarrow[C_3-7C_1]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -x^2+5x-6 & 7(x-2) \\ 0 & 0 & x-2 \end{pmatrix}$$

$$\xrightarrow[-C_2]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-5x+6 & 7(x-2) \\ 0 & 0 & x-2 \end{pmatrix} \xrightarrow[R_2-7R_3]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2-5x+6 & 0 \\ 0 & 0 & x-2 \end{pmatrix}$$

$\mapsto$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & x^2-5x+6 \end{pmatrix}$ $\leftarrow$ Smith Normal Form $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a_1 & \\ & & & \ddots \ a_n \end{pmatrix}$

$R_2 \leftrightarrow R_3$
$C_2 \leftrightarrow C_3$

invariant factors: $x-2$, x^2-5x+6.

Companion matrices: (2)
$\begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix}$

$\rightarrow$ RCF: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$.

Faster: $\text{char}(A) = (x-2)^2(x-3)$ (good for small mats.)

$\Rightarrow$ min poly$(A) = (x-2)(x-3)$
or $(x-2)^2(x-3)$.

then just check $(A-2A)(A-3I)=0$

By keeping track of the changes above... $\rightarrow$ min poly $= (x-2)(x-3) = x^2-5x+6$

can also complete change of basis matrix.

Jordan canonical form) All eigenvalues are $2, 3 \in \mathbb{Q}$.

inv. factors: $(x-2)(x-2)(x-3)$

elem divisors are $x-2$, $x-2$, $x-3$

$\Rightarrow$ JCF $= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow$ d'able.

$\boxed{\text{total } \#: q(q+1)(q-1).}$

Ex (theoretical): # Conjugacy classes in $GL_3(\mathbb{F}_q)$.

Each class rep. by one R.C.F. 3 Types: $A_1 = \begin{pmatrix} 0 & 0 & -b_0 \\ 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \end{pmatrix}$

$A_2 = \begin{pmatrix} 0 & -a_0b_0 & 0 \\ 1 & -(a_0+b_0) & 0 \\ 0 & 0 & -a_0 \end{pmatrix}$, $A_3 = \begin{pmatrix} -b_0 & 0 & 0 \\ 0 & -b_0 & 0 \\ 0 & 0 & -b_0 \end{pmatrix}$ (4)

$\det(A_1) = -b_0 \rightarrow q^2(q-1)$ total invertible

$\det(A_2) = -a_0^2 b_0 \rightarrow (q-1)^2$ total inv

$\det(A_3) = -b_0^3 \rightarrow (q-1)$ total inv