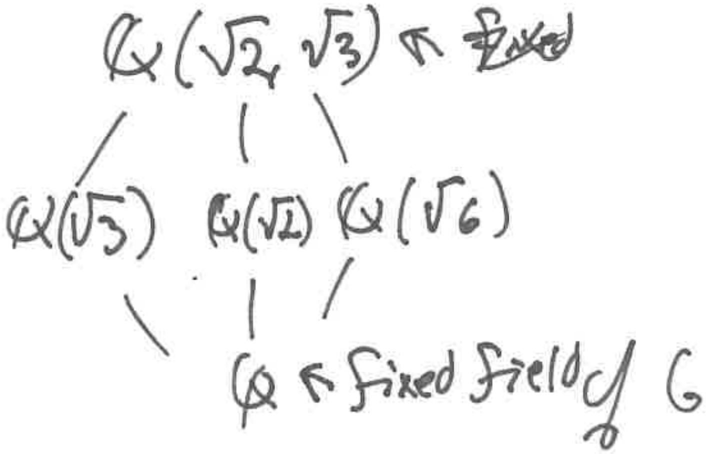
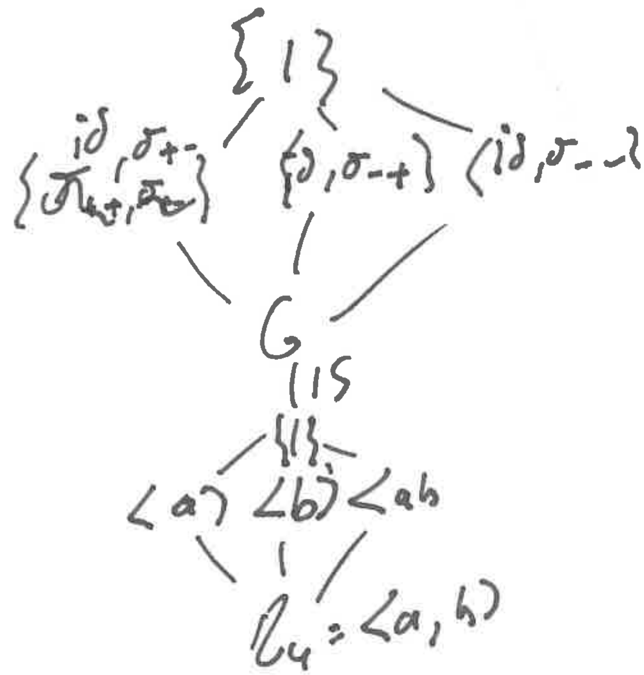
Examples: 1). $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$

We've seen: this is Galois of degree 4; call it G.

Galois group determined by 4 choices of σ: $\sqrt{2} \mapsto \pm\sqrt{2}$, $\sqrt{3} \mapsto \pm\sqrt{3}$





2). $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$. ~~not~~ not Galois

lies in $\mathbb{Q}(\sqrt[4]{2}, i) = K$ is Galois/$\mathbb{Q}$.

~~Gal/$\mathbb{Q}$~~ $\mathrm{Gal}(K/\mathbb{Q}) = \langle r, s\rangle$,

$r(\sqrt[4]{2}) = i\sqrt[4]{2}$, $r(i) = i$, $s(\sqrt[4]{2}) = \sqrt[4]{2}$, $s(i) = -i$.

(note: s is complex conj.)

Thus, $\mathrm{Gal}(K/\mathbb{Q}) \cong D_8$.

2). $\mathbb{Q}(\sqrt[4]{2}, \zeta_8)$ $^{?=16}$. $\Phi_8 = x^4+1$.

$[\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}] = [\mathbb{Q}(\zeta_8):\mathbb{Q}] = 4$.

K is splitting field of $\mathbb{Q}$ of $(x^4-2)(x^4+1)$,

so Galois

$$\begin{tikzcd}[arrows=dash]
 & K \arrow[dl, "\leq 4"'] \arrow[dr, "\leq 4"] & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[dr, "4"'] & & \mathbb{Q}(\zeta_8) \arrow[dl, "4"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$\Rightarrow [K:\mathbb{Q}] \leq 16$.

turns out, it's < 16 cuz extra alg. rel$^{\underline{n}}$.

$\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ determined by

$\sigma(\zeta_8) = \zeta_8^a$, $a \in \mathbb{Z}_8^\times$, $\sigma(\sqrt[4]{2}) = i^b \sqrt[4]{2}$.
$b \in \mathbb{Z}_4$

4 choices for a, 4 for $b \Rightarrow$ at most 16 choices

not independent as $\zeta_8 + \zeta_8^{-1} = 2\cos(\pi/4) = \sqrt{2} = (\sqrt[4]{2})^2$.

$\Rightarrow \sqrt{2} \in \mathbb{Q}(\zeta_8), \mathbb{Q}(\sqrt[4]{2})$.

$$\begin{tikzcd}[arrows=dash]
 & K \arrow[dl, "\leq 4"'] \arrow[dr, "\leq 4"] & & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[dr] & & \mathbb{Q}(\zeta_8) \arrow[dl, "2"'] \arrow[dr, "2"] & \\
 & \mathbb{Q}(\sqrt{2}) \arrow[d, "2"] & & \mathbb{Q}(i) \arrow[dll, "2"] \\
 & \mathbb{Q} & &
\end{tikzcd}$$

Now $\zeta_8 + \zeta_8^{-1} = \sqrt{2} \Rightarrow \zeta_8^2 - \sqrt{2}\,\zeta_8 + 1 = 0$

$\Rightarrow \zeta_8$ has degree ≤ 2 / $\mathbb{Q}(\sqrt[4]{2})$

$\zeta_8 \notin \mathbb{R} \Rightarrow \zeta_8 \notin \mathbb{Q}(\sqrt[4]{2}) \Rightarrow$ degree $= 2$

$\Rightarrow (K:\mathbb{Q}) = 2 \cdot 4 = 8 \Rightarrow$ the $\leq$'s in last diagram are $=$

For Galois group,

$$\sigma((\sqrt[4]{2})^2) = \sigma(\zeta_8) + \sigma(\zeta_8^{-1}),$$

$$\Rightarrow (-1)^b = \frac{\zeta_8^a + \zeta_8^{-a}}{\sqrt{2}}.$$

but $\zeta_8 = \dfrac{(1+i)}{\sqrt{2}} \overset{\text{calculation}}{\rightsquigarrow} \zeta_8^a + \zeta_8^{-a} = \sqrt{2}$

if $a \equiv 1, 7\ (8)$

$\zeta_8^a + \zeta_8^{-a} = -\sqrt{2}$ $a \equiv 3, 5 (8)$

Thus if $a \equiv 1, 7\ (8)$, $(-1)^b = 1 \Rightarrow b \equiv 0, 2\ (4)$,

$a \equiv 3, 5\ (8) \Rightarrow (-1)^b = -1 \Rightarrow b \equiv 1\ (4)$

New generators: $\zeta_8 = (1+i)/\sqrt{2} \Rightarrow K = \mathbb{Q}(\sqrt[4]{2}, i)$

choices send $\sqrt[4]{2}$ to any 4th root of 2, $i \mapsto \pm i$.

$\rightsquigarrow$ the Galois group is D_8.